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Eigenvalues of Nuclear Operators of Diagonal Type.

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ABSTRACT

A nuclear operator T on a Banach space is an operator admitting a representation $T = \Sigma T_n$ where each operator T_n has rank one and $\Sigma \|T_n\| < \infty$. A. Grothendieck proved that the sequence of eigenvalues, repeated according to their multiplicities, of such a nuclear operator must be square summable and that, in fact, two is the best possible exponent of summability. This work uses operators of diagonal type, i.e., operators which admit a diagonal representation with respect to a biorthogonal system, to construct examples of eigenvalue behavior. In particular we show that all of the known results concerning eigenvalue summability of nuclear operators can be obtained using operators of diagonal type.

In Chapter 1, we present the terminology and notation to be used in this work. Chapter 2 gives a summary of results concerning nuclear (trace class) operators on Hilbert space. In Chapter 3, some of the important known results concerning eigenvalue behavior of nuclear operators on general Banach spaces are presented.

The convolution operators on $L_p(0,1)$, $1 < p < \infty$,

$p \neq 2$, are diagonal operators since the trigonometric system $(e^{2\pi inx})$ is a conditional basis of these spaces. In Chapter 4 we use these important operators to exhibit extremal behavior of eigenvalue summability of nuclear operators on the L_p-spaces.

In Chapter 5, we study nuclear cyclic diagonal operators. If $X_1, \ldots, X_n$ are sequence spaces, an operator D on $\oplus X_i$ is a cyclic diagonal operator if its restriction to each X_i is a diagonal operator into X_{i+1} (if $i = n$ into X_1). Certain eigenspaces of these operators have important applications to Banach space theory.

Finally, in Chapter 6, we use a recently discovered space of G. Pisier to answer affirmatively a long outstanding question of Pełczyński and Saphar which is converse to Grothendieck's result: Given a (nonzero) sequence (λ_n) in ℓ_2, is there a Banach space X and a nuclear operator on X whose eigenvalue sequence is (λ_n)?

INTRODUCTION

The purpose of this work is to use operators which have a diagonal representation with respect to a biorthogonal system in order to study the summability properties of eigenvalue sequences of nuclear operators on an arbitrary Banach space.

A nuclear operator T from a Banach space X into a Banach space Y is an operator admitting a representation $T = \Sigma\, T_n$ where $\operatorname{rank}(T_n) = 1$ for each n and $\Sigma \|T_n\| < \infty$. Note that the nuclear operators are the "natural" extension of the finite rank operators and are the easiest to construct among the compact operators.

On Hilbert space, the nuclear operators coincide with the so-called trace class operators which have absolutely summable eigenvalue sequences. In the early 1950's Grothendieck, and independently Ruston, expanded the definition of the trace class operators to nuclear operators between arbitrary Banach spaces. In addition, Grothendieck showed that eigenvalue sequences of nuclear operators are square summable and that, in general, two is the best possible summability exponent. This led to a problem first posed by Pełczyński and Saphar, namely: Which (nonzero) sequences in ℓ_2 are eigenvalue sequences of nuclear operators on some Banach space?

For the study of eigenvalue distributions, those operators which are diagonal with respect to some biorthogonal system are the most natural to investigate since their defining diagonal elements are eigenvalues. This work is to show that all of the known results concerning eigenvalue distributions can be obtained from diagonal type maps.

The important convolution operators on L_p spaces are in this category, and we show in Chapters 3 and 4 that the known results concerning summability of eigenvalues of nuclear operators on these spaces can be obtained from such operators. Prior to this the examples illustrating the summability limits of eigenvalue sequences on L_p spaces were obtained by "ad hoc" methods and were far from being diagonals.

If $X_1, \ldots X_n$ are Banach spaces with bases, and D is an operator on $\oplus X_i$ whose restriction to each X_i is diagonal into X_{i+1} (for i = n the restriction is diagonal into X_1) with respect to the corresponding bases, we call D a cyclic diagonal operator. In Chapter 5 we examine nuclear cyclic diagonal operators which have applications to Banach space theory. In particular we present an alternate approach to the generation of uncomplemented subspaces of $\ell_p \oplus \ell_q$ which have certain properties.

Finally, in Chapter 6 we use a recently discovered space of Pisier to solve the problem of Pełczyński and Saphar. We present a converse to the original theorem of Grothendieck concerning eigenvalue summability of nuclear operators: There exists a Banach space P such that for every (nonzero) sequence $(\lambda_n) \in \ell_2$, there is a nuclear operator of diagonal type on P with eigenvalue sequence exactly (λ_n).

CHAPTER 1

DEFINITIONS AND STANDARD CONCEPTS

The spaces considered in this work are Banach spaces, i.e., complete normed linear spaces. An important example is Hilbert space (a complete inner product space) which will usually be denoted by H. Since our primary interest will be eigenvalue distributions, the scalar field for all spaces will be $\mathbb{C}$, the complex numbers, unless otherwise noted.

By _operator_ or _mapping_ we mean a bounded linear transformation. The collection of all operators from a space X into a space Y is denoted by $\mathfrak{L}(X,Y)$ (if $X = Y$, this will be shortened to $\mathfrak{L}(X)$). The linear space $\mathfrak{L}(X,Y)$ is a Banach space with respect to the norm

$$\|T\| := \sup_{\|x\| \le 1} \|Tx\| , \qquad T \in \mathfrak{L}(X,Y).$$

(The symbol ":=" will be used throughout to denote that the lefthand expression is to be defined by the expression on the right).

By an _isomorphism_ we mean an open one-to-one mapping. An _isometry_ T is an isomorphism with $\|Tx\| = \|x\|$ for all x. If X is a Banach space, a _projection_ P is an operator on X such that $P^2 = P$. A _subspace_ is a closed linear subset. A subspace Y is <u>complemented</u> in a space X if there is a projection $P \in \mathcal{L}(X)$ such that $P(X) = Y$.

The space $\mathcal{L}(X,\mathbb{C})$ is called the <u>dual space</u> of X and is denoted by X'. A space X is <u>reflexive</u> if the evaluation map $\phi: X \to X'' = (X')'$, defined by the relation

$$(\phi(x))(f) := f(x), \qquad x \in X,\ f \in X',$$

is onto. If $T \in \mathcal{L}(X,Y)$, then the <u>adjoint</u> of T, denoted by T^*, is that element of $\mathcal{L}(Y',X')$ defined by the relation

$$(T^*f)(x) := f(Tx), \qquad x \in X,\ f \in Y'.$$

If $\{x_\alpha\} \subset X$, then we denote by $[x_\alpha]$ the closed linear span of $\{x_\alpha\}$ in X. If (x_n) is a sequence of elements of X (we will use the informal notation $(x_n) \subset X$), and $(f_n) \subset X'$, then the paired sequence (x_n,f_n) is a <u>biorthogonal system</u> if $f_m(x_n) = \delta_{mn}$ for all $m,n \in \mathbb{N}$, the natural numbers. By a <u>basis</u> of a space X we will always mean a <u>Schauder basis</u>, that is, a biorthogonal system (x_n,f_n) such that for all $x \in X$,

$$x = \sum_{n=1}^{\infty} f_n(x)x_n,$$

where convergence is in the norm of X. A sequence (x_n) is called a <u>basic sequence</u> if it is a basis for $[x_n]$. There is a simple and useful criterion for checking if a given sequence is a basic sequence (see, e.g., [29]).

<u>Proposition 1.1</u>. Let (x_n) be a sequence of vectors in a Banach space X. Then (x_n) is a basic sequence if and only if $x_n \neq 0$ for all n and there is a constant K such that, for every choice of scalars (a_n) and integers $M < N$, we have

$$\left\| \sum_{n=1}^{M} a_n x_n \right\| \leq K \left\| \sum_{n=1}^{N} a_n x_n \right\|.$$

A basic sequence (x_n) is <u>unconditional</u> if for every scalar sequence (a_n), the convergence of $\Sigma\, a_n x_n$ implies the convergence of $\Sigma\, b_n x_n$ for any sequence (b_n) with $|b_n| \leq |a_n|$ for all n. For a proof of the following fundamental result concerning unconditional bases see, e.g., Lindenstrauss and Tzafriri, [15], p. 18.

<u>Proposition 1.2</u>. Suppose (x_n) is an unconditional basis for a Banach space X. Then there exists a constant K such that for every subset σ of the positive integers there is a projection $P_\sigma \in \mathcal{L}(X)$ onto $[x_i \colon i \in \sigma]$ with $\|P_\sigma\| \leq K$.

Spaces with bases are examples of an important class of Banach spaces. A Banach space X is said to have the <u>approximation property</u> (A.P.) if, for every Banach space Y, every compact operator $T \in \mathcal{L}(Y,X)$, and every $\varepsilon > 0$, there is an operator $T_0 \in \mathcal{L}(Y,X)$ of finite rank such that

$$\|T - T_0\| < \varepsilon.$$

If $T \in \mathcal{L}(X)$, we call T an <u>operator of diagonal type</u>

if there is a biorthogonal sequence (x_n, f_n) and scalars (λ_n) such that T has a representation

$$Tx = \sum_{n=1}^{\infty} \lambda_n f_n(x) x_n, \qquad x \in X.$$

If (x_n) is a basis for X, then T is called a <u>diagonal operator</u>.

We will use the following classical Banach spaces extensively:

For $1 \leq p \leq \infty$, denote by ℓ_p the Banach space of scalar sequences $a = (a_n)$ which are absolutely p^{th} power summable (resp. bounded for $p = \infty$) with norm

$$\|a\|_p = \begin{cases} \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}, & 1 \leq p < \infty \\ \sup_n |a_n| & p = \infty. \end{cases}$$

The subspace of ℓ_∞ consisting of all null sequences is denoted by c_0. By ℓ_p^n, $1 \leq p \leq \infty$, $n \in N$, we mean the n dimensional analog of ℓ_p.

For a positive measure μ on a measure space Ω, denote by $L_p(\Omega,\mu)$, $1 \leq p \leq \infty$, the Banach space of p^{th} power integrable functions $f\colon \Omega \to \mathbb{C}$, (resp. equivalence classes of essentially bounded measurable functions for $p = \infty$) with norm

$$\|f\|_p = \begin{cases} \left(\int_\Omega |f|^p d\mu\right)^{1/p} & 1 \leq p < \infty \\ \operatorname{ess\,sup}_{x\in\Omega} |f(x)| & p = \infty. \end{cases}$$

If μ is Lebesgue measure on some measurable subset A of R^n, then we will frequently employ the notation $L_p(A)$.

If (X_n) is a (finite or infinite) sequence of Banach spaces, we will use the notation $(\oplus X_n)_p$, $1 \leq p \leq \infty$, to mean the completion of the direct sum of the spaces X_n under the norm

$$\|(x_n)\|_p = \begin{cases} (\Sigma \|x_n\|^p)^{1/p} & 1 \leq p < \infty \\ \sup_n \|x_n\| & p = \infty. \end{cases}$$

Operator Ideals

The subject of this work, Grothendieck's class of nuclear operators, is an example of a far more general concept which has become an extremely useful tool in Banach space theory.

If X and Y are Banach spaces, then for $f \in X'$ and $y \in Y$ we will use the tensor notation $f \otimes y$ to mean the rank one operator from X into Y defined by

$$(f \otimes y)(x) := f(x)y, \qquad x \in X.$$

Using this notation, we make the following definitions.

Definition 1.3. Let $\mathcal{L}$ denote the class of all operators between arbitrary Banach spaces. A class $\mathcal{A} \subset \mathcal{L}$ of operators is an operator ideal if for each set $\mathcal{A}(X,Y) = \mathcal{A} \cap \mathcal{L}(X,Y)$ one has

i. $f \otimes y \in \mathcal{A}(X,Y)$ for all $f \in X'$, $y \in Y$,

ii. If $S, T \in \mathcal{A}(X,Y)$ and $c \in \mathbb{C}$, then $cS + T \in \mathcal{A}(X,Y)$,

iii. If $S \in \mathcal{A}(X,Y)$, $R \in \mathcal{L}(Y,Z)$ and $T \in \mathcal{L}(W,X)$, then $RST \in \mathcal{A}(W,Z)$.

Thus in particular an operator ideal restricted to a single space X is a two-sided ideal in the ring of operators on X which contains (by i. and ii.) all of the finite rank operators. For obvious reasons, property iii. is called the "ideal property."

Definition 1.4. Let $\mathcal{A}$ be an operator ideal. A non-negative function $\alpha: \mathcal{A} \to R$ is an ideal norm if for all Banach spaces W, X, Y, Z,

i. $\alpha(f \otimes y) = \|f\|\|y\|$ for all $f \in X'$, $y \in Y$,

ii. If $S, T \in \mathcal{A}(X,Y)$, then $\alpha(S + T) \leq \alpha(S) + \alpha(T)$,

iii. If $R \in \mathcal{L}(Y,Z)$, $S \in \mathcal{A}(X,Y)$ and $T \in \mathcal{L}(W,X)$, then $\alpha(RST) \leq \|R\|\alpha(S)\|T\|$.

If $\mathcal{A}$ possesses an ideal norm α, then the pair $(\mathcal{A},\alpha)$ is called a normed ideal. If, in addition, each component $\mathcal{A}(X,Y)$ of $\mathcal{A}$ is complete under the norm α, then $(\mathcal{A},\alpha)$ is called a Banach ideal.

The following operator ideals are used in this work.

The finite rank operators, denoted by $\mathcal{F}$ form a normed ideal (but not a Banach ideal) under the operator norm. This ideal consists of all operators admitting a representation

$$T = \sum_{i=1}^{n} f_i \otimes y_i,$$

where $f_1, \ldots, f_n \in X'$, $y_1, \ldots, y_n \in Y$, $n \in N$, and X

and Y are arbitrary Banach spaces.

The <u>closure of the finite rank operators</u>, denoted by $\overline{\mathcal{F}}$, consists of those operators which are the limits of finite rank operators in the operator norm. Thus $(\overline{\mathcal{F}}, \|\cdot\|)$ is a Banach ideal.

The <u>nuclear operators</u> are denoted by $\mathcal{N}$. In tensor notation, each component $\mathcal{N}(X,Y)$ consists of those operators T which admit a representation

$$T = \sum_{n=1}^{\infty} f_n \otimes y_n, \qquad (f_n) \subset X', \ (y_n) \subset Y,$$

such that

$$\sum_{n=1}^{\infty} \|f_n\| \|y_n\| < \infty.$$

Any such representation is called a nuclear representation of T. The infimum of $\Sigma\|f_n\|\|y_n\|$ over all nuclear representations of T is denoted by $\nu(T)$. It is an ideal norm and $(\mathcal{N}, \nu)$ is a Banach ideal. The ideal $\mathcal{N}$ has the extremely important property that if $\mathcal{Q}$ is any Banach ideal, then $\mathcal{N} \subset \mathcal{Q}$.

The ideal of <u>compact operators</u>, denoted by $\mathcal{K}$, consists of all operators possessing the property that the image of any bounded set is precompact. $(\mathcal{K}, \|\cdot\|)$ is a Banach ideal.

For $1 \leq p < \infty$, the <u>absolutely p-summing operators</u>, denoted by Π_p, consist of those operators which map weakly p^{th} power summable sequences into absolutely p^{th} power summable sequences. I.e., $T \in \Pi_p(X,Y)$ if there exists a

$c > 0$ such that

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{1/p} \leq c \sup_{\|f\| \leq 1} \left(\sum_{i=1}^{n} |f(x_i)|^p\right)^{1/p}$$

for all $x_1, \ldots, x_n \in X$. If we denote

$$\pi_p(T) := \inf c$$

where c satisfies the above inequality, then (Π_p, π_p) is a Banach ideal.

We denote by $\mathcal{I}_p$, $1 \leq p < \infty$, the ideal of <u>p-integral operators</u>. An operator T is in $\mathcal{I}_p(X,Y)$ if there is a probability space (Ω,μ) and operators $R \in \mathcal{L}(X, L_\infty(\Omega,\mu))$ and $S \in \mathcal{L}(L_p(\Omega,\mu), Y'')$ such that $SiR = \phi T$ where i is the inclusion of $L_\infty(\Omega,\mu)$ into $L_p(\Omega,\mu)$ and ϕ is the canonical evaluation map of Y into Y''. If

$$\iota_p(T) := \inf \|R\| \|S\|$$

where the infimum is taken over all such factorizations, then $(\mathcal{I}_p, \iota_p)$ is a Banach ideal.

An important property of the ideal of compact operators on a space X is that the spectrum of any $T \in \mathcal{K}(X)$ is a null sequence and each non-zero element of that spectrum is an eigenvalue [1]. We may therefore make the following definition.

<u>Definition 1.5</u>. Let X be a Banach space and $T \in \mathcal{K}(X)$. An <u>eigenvalue sequence</u> of T, denoted by $(\lambda_n(T))$, is

an ordering of the non-zero eigenvalues of T, each repeated according to its (finite) multiplicity.

Remark. The more standard definition of an eigenvalue sequence specifies a decreasing order, i.e., $|\lambda_n(T)| \geq |\lambda_{n+1}(T)|$ for all n. Unless a particular ordering is specified, either definition may be applied. We will have occasion, however, to specify orderings which may not be decreasing when questions of conditional convergence arise.

CHAPTER 2

TRACE CLASS OPERATORS ON HILBERT SPACE

The study of the summability properties of eigenvalue sequences of operators on Hilbert space historically has been closely tied to the functional trace. Recall that if T is an operator on a finite dimensional space with representation

$$T = \sum_{i=1}^{n} f_i \otimes x_i,$$

then the number

$$\text{(1)} \qquad \text{Functional Trace } (T) = \sum_{i=1}^{n} f_i(x_i)$$

is independent of representation. On the other hand we also have that

$$\text{(2)} \qquad \text{Spectral Trace } (T) = \sum_{i=1}^{m} (\lambda_i(T)).$$

It has long been known that (1) and (2) have the same value for any finite dimensional operator T, so we may unambiguously speak of the "trace" of such an operator.

In 1936, E. J. Murray and J. von Neumann [19] investigated the problem: What class of operators on infinite dimensional Hilbert space have a well-defined and continuous trace? Their investigation led to the following definition.

<u>Definition 2.1</u>. An operator $T \in \mathcal{L}(H)$ is said to be a <u>trace class operator</u> if there exist Hilbert-Schmidt operators (defined below) A and B on H such that $T = AB$.

It turns out that such an operator T is what we now call a nuclear operator. Thus it has a representation

$$T = \sum_{n=1}^{\infty} f_n \otimes x_n \qquad \text{with} \quad \sum_{n=1}^{\infty} \|f_n\| \, \|x_n\| < \infty.$$

Moreover, on Hilbert space in this case we may define a continuous linear functional on this trace class

$$\text{Functional Trace } (T) = \sum_{n=1}^{\infty} f_n(x_n),$$

the sum being independent of representation by continuity considerations. Unfortunately, the question of whether (2) could be extended to this trace class of operators was more difficult. Finally, in 1959, Lidskij [14] proved the following result.

<u>Theorem 2.2</u>. Let T be a trace class operator on a Hilbert space H. Then

$$\text{Functional Trace } (T) = \sum_{n=1}^{\infty} \lambda_n(T).$$

Thus we may unambiguously write Trace (T) for operators T in the trace class.

Meanwhile, Von Neumann and R. Schatten [20] formulated in 1946 the modern definition of the Hilbert-Schmidt operators and extended it to a much broader collection of operator classes. Recall that if $T \in \mathcal{L}(H)$, then TT^* is a positive operator and so we may define the operator $A = \sqrt{TT^*}$ to be that positive operator such that $A^2 = TT^*$.

Definition 2.3. Let $0 < p < \infty$. An operator $T \in \mathcal{L}(H)$ belongs to the Schatten p-class of operators $\mathcal{S}_p$ if

$$\sum_{n=1}^{\infty} (\lambda_n(\sqrt{TT^*}))^p < \infty.$$

For $p = 2$ we have exactly the Hilbert-Schmidt operators. Of more importance to our study, the trace class operators belong to the class $\mathcal{S}_1$, and the trace class (nuclear) norm is given by

$$\nu(T) = \sum_{n=1}^{\infty} \lambda_n(\sqrt{TT^*}).$$

Finally, the relationship between the summability of $(\lambda_n(TT^*))$ and of $(\lambda_n(T))$ was discovered by H. Weyl [31] in 1949.

Theorem 2.4. (Weyl Inequality - Additive Form).
Let $T \in \mathcal{L}(H)$ and $0 < p < \infty$. Then

$$\sum_{n=1}^{\infty} |\lambda_n(T)|^p \leq \sum_{n=1}^{\infty} (\lambda_n(\sqrt{TT^*}))^p.$$

In particular, if $T \in \mathcal{S}_p$, then the eigenvalues of T are absolutely p^{th} power summable.

Thus on Hilbert space the summability properties of eigenvalue sequences of nuclear operators are quite simple: Every nuclear operator has absolutely summable eigenvalues. Since diagonal operators with respect to, e.g., the unit vector basis of ℓ_2 defined by an ℓ_1 sequence are clearly nuclear, every (nonzero) absolutely summable sequence is the eigenvalue sequence of a nuclear operator.

We close this chapter with a final remark that will be used in the sequel. On Hilbert space, just as the ideal of nuclear operators coincides with the trace class operators, the absolutely p-summing operators ($1 \leq p < \infty$) all coincide with the Hilbert-Schmidt operators [21].

CHAPTER 3

SOME KNOWN RESULTS ON BANACH SPACES

We present in this chapter a brief summary of the major results in the study of eigenvalues of nuclear operators. In the three remaining chapters we will focus our attention on a specific class of nuclear operators, those of diagonal type. We will see that this involves no real loss of generality and, in fact, allows us to answer several questions concerning eigenvalue behavior.

We have seen that eigenvalue sequences of trace class operators on Hilbert space must be absolutely summable, and that every (nonzero) absolutely summable sequence is an eigenvalue sequence of a trace class operator. The corresponding questions for Banach spaces are: "What are the summability properties of eigenvalue sequences of nuclear operators on Banach spaces?" and "Which sequences can actually be realized as eigenvalue sequences of nuclear operators?" The latter of these questions was first posed by Pełczyński and Saphar and will be taken up in Chapter 6. Grothendieck gave an answer to the former of these questions which sharply separated the situation on general Banach spaces from that on Hilbert space. Prior to presenting this result we first prove a proposition also due to Grothendieck [5] .

Propcsition 3.1. Let X be a Banach space and $T \in \mathcal{N}(X)$. Then T has a factorization

$$T: X \xrightarrow{A} c_0 \xrightarrow{D} \ell_1 \xrightarrow{B} X$$

where A and B are compact and D is a nuclear operator which is diagonal relative to the unit bases of c_0 and ℓ_1.

Proof. We will use the following lemma concerning "factorization" of sequences.

Lemma 3.2. Let $(\lambda_n) \in \ell_1$. Then there exist sequences $(\alpha_n) \in c_0$ and $(\mu_n) \in \ell_1$ such that $\lambda_n = \alpha_n \mu_n$ for all n. Moreover (α_n) can be chosen to be a positive decreasing sequence.

Proof of Lemma. Let $(N_k)_{k=1}^{\infty}$ be an increasing sequence of indices such that

$$\sum_{n=N_k+1}^{\infty} |\lambda_n| < \frac{1}{k \cdot 2^k}, \qquad k \in \mathbb{N}.$$

Let $\sigma_k := \{N_k+1, \ldots, N_{k+1}\}$ and

$$\alpha_n = \begin{cases} 1 & n \leq N_1 \\ 1/k & n \in \sigma_k. \end{cases}$$

Then $(\alpha_n) \in c_0$ and if $\mu_n := \lambda_n/\alpha_n$, $n = 1,2,\ldots$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} |\mu_n| &= \sum_{n=1}^{N_1} |\lambda_n| + \sum_{k=1}^{\infty} \Big(\sum_{n\in\sigma_k} k|\lambda_n|\Big) \\ &= \sum_{n=1}^{N_1} |\lambda_n| + \sum_{k=1}^{\infty} k\Big(\sum_{n\in\sigma_k} |\lambda_n|\Big) \end{aligned}$$

$$\leq \sum_{n=1}^{N_1} |\lambda_n| + \sum_{k=1}^{\infty} k\left(\frac{1}{k\cdot 2^k}\right) < \infty.$$

So $(\mu_n) \in \ell_1$ which proves the lemma.

To prove the proposition, since T is nuclear it has a representation

$$T = \sum_{n=1}^{\infty} \lambda_n f_n \otimes x_n,$$

where $(\lambda_n) \in \ell_1$ and (f_n), (x_n) are normalized sequences in X' and X respectively. Applying Lemma 3.2 twice we can find sequences (α_n), $(\beta_n) \in c_0$ and $(\mu_n) \in \ell_1$ such that $\lambda_n = \alpha_n \mu_n \beta_n$, $n = 1,2,\ldots$.

Define

$$A: X \to c_0 \quad \text{by} \quad Ax = (\alpha_n f_n(x))_{n=1}^{\infty}, \quad x \in X,$$

$$D: c_0 \to \ell_1 \quad \text{by} \quad D(a_n) = (\mu_n a_n), \quad (a_n) \in c_0,$$

$$B: \ell_1 \to X \quad \text{by} \quad B(a_n) = \sum_{n=1}^{\infty} \beta_n a_n x_n, \quad (a_n) \in \ell_1.$$

The operator D is nuclear as illustrated by its natural diagonal representation. Moreover, both A and B are elements of $\overline{\mathcal{F}}$, and hence compact. Since for all $x \in X$,

$$BDAx = \sum_{n=1}^{\infty} \alpha_n \mu_n \beta_n f_n(x) x_n = \sum_{n=1}^{\infty} \lambda_n f_n(x) x_n = Tx,$$

we have the desired factorization.

We can now give a proof using related operators of Grothendieck's theorem on the summability of eigenvalue sequences of nuclear operators. If $A \in \mathcal{L}(X,Y)$ and

$B \in \mathcal{L}(Y,X)$ then AB and BA are called related operators. (Pietsch, [22]). Every nonzero eigenvalue of finite multiplicity of AB is also an eigenvalue of BA and has the same multiplicity. Thus the eigenvalue sequences of related compact operators coincide.

<u>Theorem 3.3</u>. (Grothendieck [5]). Let X be a Banach space and $T \in \mathcal{L}(X)$. Then

$$(\lambda_n(T)) \in \ell_2.$$

Moreover, two is the best possible exponent of convergence.

<u>Proof</u>. Factor $T = BDA$ as in Proposition 3.1. The diagonal operator D defined by the sequence $(\mu_n) \in \ell_1$ can be further factored

$$D\colon c_0 \xrightarrow{D_1} \ell_2 \xrightarrow{D_2} \ell_1$$

where $D_1(a_n) = (\mu_n a_n)$ and $D_2(b_n) = (\mu_n b_n)$. Since

$$\|D_1(a_n)\|_2 = \Big(\sum_{n=1}^{\infty} |\mu_n||a_n|^2\Big)^{\frac{1}{2}} \leq (\sup_n |a_n|)\Big(\sum_{n=1}^{\infty} |\mu_n|\Big)^{\frac{1}{2}},$$

and

$$\|D_2(b_n)\|_1 = \sum_{n=1}^{\infty} |\mu_n||b_n| \leq \Big(\sum_{n=1}^{\infty} |\mu_n|\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^{\infty} |b_n|^2\Big)^{\frac{1}{2}},$$

both D_1 and D_2 are continuous.

We now have the factorization

$$T\colon X \xrightarrow{A} c_0 \xrightarrow{D_1} \ell_2 \xrightarrow{D_2} \ell_1 \xrightarrow{B} X,$$

and from the above remark concerning related operators,

$$(\lambda_n(T)) = (\lambda_n(BD_2D_1A)) = (\lambda_n(D_1ABD_2)).$$

Thus, by the Weyl inequality, to prove that $(\lambda_n(T)) \varepsilon \ell_2$ it suffices to show that $D_1ABD_2: \ell_2 \to \ell_2$ is a Hilbert-Schmidt operator.

A well-known result due to Grothendieck in the theory of absolutely p-summing operators (see, e.g., [22] , p. 307) states that

$$\Pi_2(c_0,\ell_2) = \mathcal{L}(c_0,\ell_2).$$

So $D_1 \varepsilon \Pi_2(c_0,\ell_2)$, and by the ideal property, $D_1ABD_2 \varepsilon \Pi_2(\ell_2)$. But then D_1ABD_2 is Hilbert-Schmidt since on Hilbert space the absolutely 2-summing and Hilbert-Schmidt operators coincide.

We have left to show that two is the optimum summability exponent. Although there are several simple ways to do this (see the remark following this proof) we will reproduce Grothendieck's original example. Not only does this example involve an operator of diagonal type, which is the central topic of this work, but also we will use similar operators in Chapter 4 to illustrate other important aspects of eigenvalue behavior.

We must exhibit a nuclear operator T such that $(\lambda_n(T)) \varepsilon \ell_2$ but $(\lambda_n(T)) \not\varepsilon \ell_p$ for any $p < 2$. Since our operator will involve convolution and fourier series we will identify $L_1(0,1)$ with $L_1(\mathbb{T})$ where $\mathbb{T}$ is the unit circle, thus extending all functions with period 1. Let

$$f(x) = \sum_{n=2}^{\infty} \frac{e^{in\mathrm{Log}\, n}}{n^{\frac{1}{2}}(\mathrm{Log}\, n)^{\beta}} e^{2\pi inx}, \qquad \beta > 1.$$

It is shown in Zygmund [32] that f is continuous. Define the convolution operator $T_f: L_1(0,1) \to L_1(0,1)$ by

$$(1) \qquad T_f g(x) = f*g(x) = \int_0^1 f(x-y)g(y)\,dy, \qquad g \in L_1(0,1).$$

It is well known (see, e.g., [32]) that the eigenvalues of a convolution operator such as is defined in (1) are exactly the fourier coefficients of the function f, so that in this case

$$(\lambda_n(T_f)) = (\hat{f}(n)) = \left(\frac{e^{in\mathrm{Log}\, n}}{n^{\frac{1}{2}}(\mathrm{Log}\, n)^{\beta}} \right).$$

Thus it is apparent that $(\lambda_n(T_f)) \in \ell_2$ but $(\lambda_n(T_f)) \notin \ell_p$ for any $p < 2$.

Grothendieck proved that _if_ $T_f \in (L_1(0,1))$ _is defined as in_ (1) _by a continuous function_ f, _then_ T_f _is nuclear_. We omit the proof of this result, but remark that since $\|f*g\|_\infty \leq \|f\|_\infty\|g\|_1$, T_f factors thru $L_\infty(0,1)$ and so is integral by definition. Grothendieck actually showed that any operator factorable into an integral operator followed by a weakly compact operator is nuclear, then exhibited such a factorization for T_f.

Since f is continuous, T_f is nuclear and has eigenvalues with the required summability properties. This completes the proof.

Remark. As was mentioned in the above proof there are several alternate methods of constructing a nuclear operator whose eigenvalue sequence is square summable but no better. For example, let $(\lambda_n) \in \ell_2$ and $D: \ell_2 \to \ell_2$ be a diagonal map defined by the sequence (λ_n). Let $\phi: \ell_2 \to \ell_\infty$ be an isometric embedding, and $X = \phi(\ell_2)$. We claim that $T = \phi D \phi^{-1}: X \to \ell_\infty$ is nuclear. To see this consider

$$T^* = (\phi^{-1})^* D^* \phi^*: \ell_\infty' \to X'.$$

Now D^* is Hilbert-Schmidt (hence in Π_2) and ϕ^* is a map from an L_1-space into a Hilbert space, hence is absolutely 2-summing (see [22]). As the composition of two absolutely 2-summing maps, T^* is thus nuclear. But if $T \in \mathcal{L}(X, \ell_\infty)$ and $T^* \in \mathcal{N}$, then $T \in \mathcal{N}$. (See, e.g., [22]).

Finally let $\Sigma f_n \otimes x_n$ be a nuclear representation of T. Since X is a subspace of ℓ_∞ let $(\bar{f}_n)$ be Hahn-Banach extensions of the functionals (f_n) to ℓ_∞. If we define

$$\bar{T} = \Sigma \bar{f}_n \otimes x_n: \ell_\infty \to \ell_\infty,$$

then $\bar{T}$ is nuclear by definition and the original sequence (λ_n) is a subsequence of its eigenvalue sequence. So $(\lambda_n(\bar{T}))$ is square summable but no better. Note however that because the functionals (f_n) were extended there is no way of determining whether $\bar{T}$ has eigenvalues in addition to the given sequence.

Similar examples can be constructed using, e.g., the span of the Rademacher functions in L_1.

The operator T_f defined in the last part of the proof of Theorem 3.3 is important to this work in two respects.

First, if we define sequences $(e_n) \in L_1$ and $(f_n) \in L_\infty = L_1'$ by

$$e_n(x) = e^{2\pi inx} \qquad x \in (0,1),\ n \in Z, \text{ and}$$

$$f_n(x) = e^{-2\pi inx} \qquad x \in (0,1),\ n \in Z,$$

then (e_n, f_n) is a biorthogonal system for L_1 (but not a basis, see Singer [29]). Moreover, T_f is diagonal with respect to this system. Thus this first example of a nuclear operator whose eigenvalue sequence is not absolutely summable is of diagonal type.

Second, if we order the eigenvalue sequence of T_f in the order corresponding to the "natural" order of the sequence $(e_n(x))_{n=-\infty}^{\infty}$, then since the fourier series for f converges uniformly (Zygmund, [32]), we have that

$$\sum_{n=-\infty}^{\infty} \lambda_n(T_f) = \sum_{n=2}^{\infty} \hat{f}(n) = f(0),$$

so this eigenvalue sequence is conditionally convergent.

Theorem 3.3 implies that the eigenvalue sequences of nuclear operators on an arbitrary Banach space may not be absolutely summable and that this situation does in fact exist, at least on the space L_1. A result of Johnson, König, Maurey and Retherford shows that there are nuclear operators with non-absolutely summable eigenvalues on every infinite dimensional Banach space except Hilbert space.

Theorem 3.4. Suppose every nuclear operator on a Banach space X has an absolutely summable eigenvalue sequence. Then X is isomorphic to a Hilbert space.

See [10], pp. 376-378 for a proof.

We thus have the extreme situations of square summability of eigenvalue sequences on L_1 and absolute summability on L_2. Much work has been done in determining orders of summability on other spaces and classes of spaces. We end this chapter by presenting, without proof, two of the more important results in this area. The first generalizes the L_1 and L_2 cases to all of the L_p's, $1 \leq p \leq \infty$.

Theorem 3.5. (Johnson, König, Maurey, Retherford [10]) Let $1 \leq p \leq \infty$ and $1/q = 1 - |1/2 - 1/p|$. Assume (Ω,μ) is a measure space and $T \in \mathcal{N}(L_p(\Omega,\mu))$. Then $(\lambda_n(T)) \in \ell_q$.

Theorem 3.5 interpolates between our previous results, for if $p = 1$ in the theorem then $q = 2$, while if $p = 2$ then $q = 1$.

The following example (also from [10]) shows that the summability order q of Theorem 3.5 cannot be lowered.

Example 3.6. Define the Littlewood matrices inductively.

$$A_{2^0} = (1), \qquad A_{2^n} = \begin{pmatrix} A_{2^{n-1}} & A_{2^{n-1}} \\ A_{2^{n-1}} & -A_{2^{n-1}} \end{pmatrix} \qquad n \in \mathbb{N}.$$

We can derive from the inductive process that $A^2_{2^n} = 2^n \mathrm{Id}$, and moreover that for each eigenvalue λ of $A_{2^{n-1}}$ and corresponding eigenvector x, the vectors $(x,(1-\sqrt{2})x)$ and $(x,(-1-\sqrt{2})x)$ are eigenvectors of A_{2^n} with eigenvalues $\sqrt{2}\lambda$ and $-\sqrt{2}\lambda$ respectively. Thus the eigenvalues of A_{2^n} are $\pm 2^{n/2}$, each having multiplicity 2^{n-1}.

For $1 \le p \le 2$, $1/p + 1/p' = 1$, define

$$A = \sum_{n=1}^{\infty}\oplus\, n^{-2}(2^n)^{-(1+1/p')}A_{2^n}\colon (\oplus \ell_p^{2^n})_p \to (\oplus \ell_p^{2^n})_p,$$

so that $A\colon \ell_p \to \ell_p$ is a block sum of multiples of the matrices A_{2^n}. Denote by $A_{2^n}^j$ the j^{th} row of A_{2^n}, which consists of ± 1's. If we consider $A_{2^n}^j = (a_{ij})_{i=1}^{2^n}$ as an element of $\ell_{p'}^{2^n} = (\ell_p^{2^n})'$ with action

$$A_{2^n}^j(x_i) = \sum_{i=1}^{2^n} a_{ij}x_i, \qquad (x_i) \in \ell_p^{2^n},$$

then

$$A_{2^n} = \sum_{j=1}^{2^n} A_{2^n}^j \otimes e_j,$$

where (e_j) is the unit vector basis of $\ell_p^{2^n}$. Thus

$$\nu(A_{2^n}) \le \sum_{j=1}^{2^n} \|A_{2^n}^j\|_{p'}\,\|e_j\|_p = 2^n.2^{n/p'} = 2^{n(1+1/p')},$$

and so

$$\nu(A) \le \sum_{n=1}^{\infty} n^{-2}(2^n)^{-(1+1/p')}\nu(A_{2^n}) \le \sum_{n=1}^{\infty} n^{-2} < \infty.$$

Therefore $A \in \mathfrak{N}(\ell_p)$. Moreover the eigenvalues of A are exactly the eigenvalues of the multiples of the A_{2^n}'s. So taking into account their multiplicities we have

$$\sum_{j=1}^{\infty} |\lambda_j(A)|^q = \sum_{n=1}^{\infty} 2^n\{n^{-2q}(2^n)^{-q(1+1/p')}(2^{n/2})^q\}$$

$$= \sum_{n=1}^{\infty} n^{-2q} \cdot 2^{n(1-(1/2+1/p'))}.$$

This last expression is finite if and only if

$$1 - q(1/2 + 1/p') \leq 0, \quad \text{i.e.,} \quad 1/q \leq 1 - |1/2 - 1/p|$$

Thus we have constructed nuclear operators $A: \ell_p \to \ell_p$, for each $1 \leq p \leq 2$, such that $(\lambda_n(A)) \in \ell_q$ but $(\lambda_n(A)) \notin \ell_r$ for any $r < q$. Moreover $A^*: \ell_{p'} \to \ell_{p'}$ is nuclear and has the same spectrum as A. Since $1-|1/2-1/p'| = 1-|1/2-1/p|$, the operators A and A^* on the spaces $\ell_p = L_p(\mathbb{N}, \text{counting measure})$, $1 \leq p \leq \infty$, show that the exponent q in Theorem 3.5 cannot be improved.

The next theorem, due to König, Retherford and Tomczak-Jaegermann [13], gives bounds on the summability of eigenvalues of nuclear operators on certain classes of Banach spaces. We need the following definition.

Definition 3.7. Let $1 < p \leq 2$. We say that a Banach space X is of (Rademacher) type p if there is a constant $c > 0$ such that for every finite sequence $(x_n) \subset X$ we have

$$\left(\int_0^1 \|\sum_n r_n(t)x_n\|^2 dt\right)^{1/2} \leq c\left(\sum_n \|x_n\|^p\right)^{1/p},$$

where $r_n(t) = \text{sign}(\sin 2^n \pi t)$ are the Rademacher functions on the unit interval.

Similarly, if $2 \leq q < \infty$, a space X is of (Rademacher) cotype q if there is a constant $c_1 > 0$ such that for every finite sequence $(x_n) \subset X$,

$$\left(\sum_n \|x_n\|^q\right)^{1/q} \leq c_1\left(\int_0^1 \left\|\sum_n r_n(t)x_n\right\|^2 dt\right)^{\frac{1}{2}}.$$

We note that $L_p(\Omega,\mu)$ is always of type p, cotype 2, for $1 < p \leq 2$, and of type 2, cotype p for $2 < p < \infty$. For other facts on the notions of type and cotype we refer to Maurey and Pisier [17] and Lindenstrauss and Tzafriri[15].

<u>Theorem 3.8</u>. Let X be a Banach space of type p and cotype q with $1/p - 1/q < 1/2$. Then for all $T \in \mathcal{N}(X)$,

$$(n^{1/s}\lambda_n(T)) \in \ell_\infty,$$

where $1/s = 1 - (1/p - 1/q)$.

For a proof, see [13], p. 117.

Thus for spaces satisfying the requirements of the theorem, eigenvalue sequences of nuclear operators most be "almost" in ℓ_s, i.e., in $\ell_{s+\varepsilon}$ for any $\varepsilon > 0$. Note that for the L_p-spaces, $1 < p < \infty$, this result is weaker than Theorem 3.5. At this time it is unknown whether Theorem 3.8 is sharp. In particular, is there a nuclear operator on a space X of type p and cotype q $(1/p - 1/q < 1/2)$ such that $(\lambda_n(T)) \notin \ell_s$ where $1/s = 1 - (1/p - 1/q)$? In fact, the exponent s itself may be too large. On some classes of spaces of type p and cotype q (cf. [25]) it is known that eigenvalue sequences of nuclear operators must be absolutely r-summing, where

$$1/r = 1 - \max(1/p - 1/2,\ 1/2 - 1/q).$$

This formula has the somewhat aesthetic advantage that it may apply also to the case in which $1/2 < 1/p - 1/q$. An attempt to extend Theorem 3.8 in this fashion would yield $s > 2$ which is weaker than Grothendieck's general result.

CHAPTER 4

DIAGONAL NUCLEAR OPERATORS

We here study nuclear operators on a Banach space which have a diagonal representation relative to a basis of that space. Diagonal operators are the simplest operators to define and (perhaps in consequence) the most frequently encountered operators. The major results of this chapter, Theorem 4.5 and Example 4.8, deal with convolution operators on $L_p(0,1)$, which are fundamental to large areas of both abstract and applied analysis.

If T is an operator of diagonal type on a Banach space X with diagonal representation

$$Tx = \sum_{n=1}^{\infty} \lambda_n f_n(x) x_n, \qquad x \in X,$$

then it is clear that the scalars λ_n are among the eigenvalues of T. Of most importance to this study is the case in which the sequence (λ_n) is actually the eigenvalue sequence for T. The following proposition provides two important cases in which this situation occurs.

<u>Proposition 4.1</u>. Let (x_n, f_n) be a biorthogonal system on a Banach space X such that either

(1) (x_n) is a basic sequence, or

(2) (f_n) is total on X, i.e., if $x \in X$, $x \neq 0$, then there is an $m \in N$ such that $f_m(x) \neq 0$.

Then the nonzero elements of the diagonal sequence of any operator $T \in \mathcal{L}(X)$ of diagonal type relative to (x_n, f_n) form the eigenvalue sequence of T.

Proof. If (x_n) is a basic sequence, then (f_n) is total on $[x_n]$. But each eigenvector of T is an element of $[x_n]$, so restricting T to $[x_n]$ does not change its eigenvalues. It thus suffices to consider the case in which (f_n) is total on X.

Let $\lambda \neq 0$ be an eigenvalue of T with corresponding eigenvector $x \neq 0$. Then there is an $m \in \mathbb{N}$ such that $f_m(x) \neq 0$ and

$$\lambda f_m(x) = f_m(Tx) = f_m\left(\sum_{n=1}^{\infty} \lambda_n f_n(x) x_n\right) = \lambda_m f_m(x).$$

So $\lambda = \lambda_m$. Since each nonzero λ_n is an eigenvalue $(Tx_n = \lambda_n x_n)$, the nonzero elements of (λ_n) form an eigenvalue sequence.

The simplest class of operators of diagonal type whose defining sequences are also eigenvalue sequences are the diagonal operators. Diagonal nuclear operators also have an extremely important property that carries over from the finite dimensional case. Recall that if a space X has a basis, it has the approximation property, and so $\mathcal{N}(X)$ has a continuous trace functional [5].

<u>Theorem 4.2</u>. Let X be a Banach space with basis (x_n). Suppose $T \in \mathcal{N}(X)$ is a diagonal operator with respect to (x_n) with diagonal sequence (λ_n). Then

$$\text{Trace } (T) = \sum_{n=1}^{\infty} \lambda_n.$$

<u>Proof</u>. For $x \in X$ we may write

$$Tx = \sum_{i=1}^{\infty} \lambda_i f_i(x) x_i,$$

where (f_i) is the sequence of coefficient functionals associated with the basis (x_i) (i.e., $f_i(x_j) = \delta_{ij}$ for all i, j).

Let $(u_j) \subset X'$, $(v_j) \subset X$ be such that

$$T = \sum_{j=1}^{\infty} u_j \otimes v_j$$

is a nuclear representation of T. By Lemma 3.2, since $(\|u_j\| \|v_j\|) \in \ell_1$, we may assume that $\Sigma \|u_j\| = K < \infty$ and $v_j \to 0$.

Since (x_i) is a basis, the projections

$$s_n(x) = \sum_{i=1}^{n} f_i(x) x_i, \qquad x \in X,\ n = 1,2,\ldots$$

converge in the strong operator topology to I, the identity operator on X. In particular, the sequence of operators $(I - s_n)$ converges uniformly to 0 on the precompact set $\{v_j\}$.

For fixed n consider the operator Ts_n. Since it is of finite rank,

$$\text{Trace } (Ts_n) = \sum_{i=1}^{n} \lambda_i f_i(x_i) = \sum_{i=1}^{n} \lambda_i.$$

Moreover, since T is diagonal with respect to (x_n), $Ts_n = s_nT$ and we have

$$\begin{aligned}|\text{Trace}\ (T) - \sum_{i=1}^{n} \lambda_i| &= |\text{Trace}\ (T - Ts_n)| \\ &= |\text{Trace}\ ((I - s_n)T)| \\ &= |\text{Trace}\ (\sum_{j=1}^{\infty} u_j \otimes (v_j - s_n(v_j)))| \\ &= |\sum_{j=1}^{\infty} u_j(v_j - s_n(v_j))| \\ &\leq (\sup_j \|(I - s_n)v_j\|) \sum_{j=1}^{\infty} \|u_j\| .\end{aligned}$$

But by the uniform convergence of $(I - s_n)$ on $\{v_j\}$, this last expression can be made arbitrarily small for large n, so $\Sigma\lambda_i$ converges to Trace (T).

<u>Warning</u>: The statement of Theorem 4.2 is somewhat deceptive. It is tempting to assume that $\Sigma\lambda_i f_i \otimes x_i$ is a <u>nuclear</u> representation of T (from which the conclusion would follow by definition). In particular the Theorem leaves open the possibility that $\Sigma\lambda_n$ may only be <u>conditionally</u> convergent. In fact, the major example of this chapter (see 4.7, below) will illustrate that for one extremely important type of diagonal nuclear operator, this convergence can be conditional only, and hence the diagonal representation in that case is <u>not</u> a nuclear representation.

A simple corollary relates Theorem 4.2 to the behavior of eigenvalue sequences.

<u>Corollary 4.3</u>. Let X be a Banach space. Suppose $T \in \mathcal{N}(X)$ has an eigenvalue sequence (λ_n) with corresponding eigenvectors (x_n). If (x_n) is a basis for a complemented subspace of X, then $\Sigma\, \lambda_n$ converges.

<u>Proof</u>. Consider $T_0 = T|_{[x_n]}$. Since (x_n) is a basis for $[x_n]$, T_0 is a diagonal operator with diagonal sequence (λ_n). But since $[x_n]$ is complemented in X,

$$T_0 = PTi$$

where i is the inclusion of $[x_n]$ into X and P is a projection onto $[x_n]$. So by the ideal property, T_0 is nuclear. Applying Theorem 4.2 the result follows.

<u>Remark</u>. The requirement that $[x_n]$ be complemented in X is necessary (see Example 6.4). In addition, as previously mentioned, Example 4.7 shows that (λ_n) may not be absolutely summable.

The following proposition, which has long been known, shows that eigenvalue behavior is much more straightforward in the case of nuclear operators which are diagonal with respect to an unconditional basis.

<u>Proposition 4.4</u>. Suppose X is a Banach space with unconditional basis (x_n). If T is a diagonal nuclear operator on X with respect to (x_n) defined by a sequence (λ_n), then $(\lambda_n) \in \ell_1$.

Proof. Let σ be a finite set of positive integers and P_σ the natural projection onto $[x_j: j\epsilon\sigma]$. By Proposition 1.2, since (x_n) is unconditional there is a constant K, independent of the choice of σ, such that $\|P_\sigma\| \leq K$. Hence

$$\begin{aligned} \left|\sum_{j\epsilon\sigma} \lambda_j\right| = |\text{Trace } (P_\sigma T)| &\leq \nu(P_\sigma T) \\ &\leq \|P_\sigma\| \nu(T) \\ &\leq K\ \nu(T), \end{aligned}$$

and (λ_n) is absolutely summable as required.

So in the case of nuclear operators which are diagonal with respect to an unconditional basis, the natural diagonal representation is a nuclear representation. We have already intimated, however, that this may not be true for an arbitrary basis. We now proceed to construct diagonal nuclear operators with respect to a conditional basis whose diagonal (and hence eigenvalue) sequences are not absolutely summable. In doing this we will to some extent parallel Grothendieck's example given in the proof of Theorem 3.3 above, using $L_p(0,1)$, $1 < p < \infty$, $p \neq 2$, in place of $L_1(0,1)$. Thus we will continue to identify the unit interval with the unit circle and require periodicity of functions.

For $1 < p < \infty$, $p \neq 2$, the standard trigonometric system $(e^{2\pi inx})_{n=-\infty}^{\infty}$ is a conditional basis for $L_p(0,1)$ (see, e.g., Singer [29]). As shown previously, each convolution operator on $L_p(0,1)$ defined by a function f is diagonal with respect to this basis with diagonal sequence $(\hat{f}(n))_{n=-\infty}^{\infty}$ where

$$\hat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}dx$$

are the familiar fourrier coefficients.

We first need a criterion for nuclearity of convolution operators on $L_p(0,1)$. For $f \in L_p(0,1)$, $1 \leq p \leq \infty$, and $n \in \mathbb{N}$, define

$$s_n(f)(x) := \sum_{|k|<n} \hat{f}(k)e^{2\pi ikx},$$

$$t_n(f)(x) := \sum_{|k|\geq n} \hat{f}(k)e^{2\pi ikx}, \quad \text{and}$$

$$b_{n,p}(f) := \|t_n(f)\|_p.$$

For $1 < p < \infty$, both (s_n) and (t_n) are sequences of uniformly bounded projections on L_p. Moreover, we can assume the sequence $(b_{n,p})$ decreases monotonically for each $f \in L_p$ (if necessary using an equivalent norm on L_p, see Singer [29]).

<u>Theorem 4.5</u>. Let $1 < p < 2$, $1/p + 1/p' = 1$, and suppose $f \in L_{p'}(0,1)$ with

$$(*) \qquad \sum_{n=2}^{\infty} n^{-1/p}(\mathrm{Log}\ n)\, b_{n,p'}(f) < \infty.$$

Then the convolution operator $T_f\colon L_p(0,1) \to L_p(0,1)$ defined by $T_f g = f * g$, $g \in L_p$, is nuclear.

Proof. On reflexive spaces the nuclear operators and the integral operators coincide. Moreover, a multiplication theorem for operator ideals (see Pietsch [22] p.286) asserts that $\mathcal{I}_p \circ \Pi_{p'} = \mathcal{I}_1$. But by Holder's inequality,

$$\|T_f g\|_\infty = \|f * g\|_\infty \leq \|f\|_{p'} \|g\|_p ,$$

so we may factor

$$T: L_p \xrightarrow{T_f'} L_\infty \xrightarrow{i} L_p ,$$

where i is the natural inclusion. Since $i \in \mathcal{I}_p$ by definition, to prove the theorem we have only to show that T_f', the operator T_f viewed as a mapping into L_∞, is absolutely p'-summing.

Our method will be to view T_f' in its diagonalized form

$$T_f' g = \sum_{n=-\infty}^{\infty} \hat{f}(n)\hat{g}(n)e^{2\pi inx}, \qquad g \in L_p.$$

We will break up T_f' into a block sum of finite dimensional operators on blocks of the trigonometric system $(e^{2\pi inx})$ in order to show that its absolutely p'-summing norm is finite.

For $n = 1,2,3,\ldots$, let

$$r_{2^n} := s_{2^n} - s_{2^{n-1}} ,$$

$$Y_n := r_{2^n} L_p = \left[e^{2\pi ikx}: 2^{n-1} \leq |k| < 2^n\right]_{L_p}$$

$$X_n := s_{2^n} L_\infty = \left[e^{2\pi ikx}: |k| < 2^n\right]_{L_\infty}.$$

The spaces X_n have the property (see, e.g., König [11]) that there exist isomorphisms

$$\phi_n: X_n \to \ell_\infty^{2^{n+1}-1}$$

such that for all n,

$$\|\phi_n\| \, \|\phi_n^{-1}\| \leq c_\infty \mathrm{Log}\, 2^n = c_\infty n,$$

where c_∞ is an absolute constant.

Setting

$$A_n := T_f'|_{Y_n}: Y_n \to X_n \ , \quad \text{and}$$

$$B_n := i_n A_n r_{2^n}: L_p(0,1) \to L_\infty(0,1),$$

where i_n is the natural inclusion, we have

$$T_f' = \sum_{n=0}^{\infty} B_n$$

where $B_0: L_p \to L_\infty$ is defined by $B_0 g = \hat{f}(0)\hat{g}(0)$. Moreover each B_n, $n \geq 1$, has the following factorization

$$\begin{tikzcd}
L_p \arrow[r, "B_n"] \arrow[d, "r_{2^n}"'] & L_\infty \\
Y_n \arrow[r, "A_n"] \arrow[d, "\tilde{A}_n"'] & X_n \arrow[u, "i_n"'] \\
\ell_\infty^{2^{n+1}-1} \arrow[r, "Id_n"'] & \ell_\infty^{2^{n+1}-1} \arrow[u, "\phi_n^{-1}"']
\end{tikzcd}$$

where $\tilde{A}_n = \phi_n A_n$.

Hence if $c_p = \sup_n \|r_{2^n}\|$, then

$$\begin{aligned}\pi_{p'}(T'_f) &\le \sum_{n=0}^{\infty}\pi_{p'}(B_n)\\ &\le \pi_{p'}(B_0) + \sum_{n=1}^{\infty}\|i_n\|\,\|\phi_n^{-1}\|\,\pi_{p'}(Id_n)\,\|A_n\|\,\|r_{2^n}\|\\ &\le \|f\|_{p'} + c_p\sum_{n=1}^{\infty}\|\phi_n^{-1}\|\,\pi_{p'}(Id_n)\,\|\phi_n\|\,\|A_n\|\\ &\le \|f\|_{p'} + c_p c_\infty\sum_{n=1}^{\infty} n\|A_n\|\,\pi_{p'}(Id_n).\end{aligned}$$

We therefore need bounds on $\|A_n\|$ and $\pi_{p'}(Id_n)$.

Now for all n,

$$\begin{aligned}\|A_n\| \le \|r_{2^n}T'_f\| &= \|s_{2^n}T'_f - s_{2^{n-1}}T'_f\|\\ &\le \|T'_f - s_{2^{n-1}}T'_f\| + \|T'_f - s_{2^n}T'_f\|.\end{aligned}$$

But for $k = 1,2,\ldots$, and $\|g\|_p \le 1$,

$$\begin{aligned}\|(T'_f - s_kT'_f)(g)\|_\infty &= \|t_k(f) * g\|_\infty\\ &\le \|t_k(f)\|_{p'}\|g\|_p\\ &\le b_{k,p'}(f).\end{aligned}$$

So

$$\|A_n\| \le b_{2^{n-1},p'}(f) + b_{2^n,p'}(f) \le 2b_{2^{n-1},p'}(f).$$

Our bound for $\pi_{p'}(Id_n)$ is of some importance in its own right.

<u>Lemma 4.6</u>. Let $1 < p < \infty$ and $n \in N$. Then

$$\pi_p(Id\colon \ell_\infty^n \to \ell_\infty^n) \le n^{1/p}.$$

<u>Proof of Lemma</u>. Denote by $\ell_p^n(m)$ the space C^n normed by

$$\|(x_j)\|_m = (\sum_{j=1}^{n} |x_j|^p m_j^p)^{1/p},$$

where $m = (m_j)$ is a sequence of n non-negative numbers with $\Sigma m_j^p \leq 1$. It follows from the Pietsch factorization theorem (see Pietsch [22]) that

$$\pi_p(\mathrm{Id}: \ell_\infty^n \to \ell_\infty^n) = \inf \|\mathrm{Id}: \ell_p^n(m) \to \ell_\infty^n\|$$

with the infimum taken over all sequences $m = (m_j)$.

If $m = (n^{-1/p})$, then for $\|(x_j)\|_m \leq 1$ we have $\sup |x_j| \leq n^{1/p}$, so

$$\pi_p(\mathrm{Id}: \ell_\infty^n \to \ell_\infty^n) \leq \|\mathrm{Id}: \ell_p^n((n^{-1/p})) \to \ell_\infty^n\| \leq n^{1/p}.$$

Returning to the proof of the theorem, let

$$\sigma_n = \{2^{n-1}, \ldots ,2^n - 1\}.$$

Then if

$$K_1 = \|f\|_{p'} + c_p c_\infty \|A_1\| \pi_{p'}(\mathrm{Id}_1)$$

from our previous estimate for $\pi_{p'}(T_f')$ we have

$$\pi_{p'}(T_f') \leq K_1 + c_p c_\infty \sum_{n=2}^{\infty} n(2^{n+1} - 1)^{1/p'} 2b_{2^{n-1},p'}(f)$$

$$\leq K_1 + c_p c_\infty \sum_{n=2}^{\infty} n \cdot 2^{(n+1)/p'} \cdot 2^{-n+3} \sum_{k\in\sigma_{n-1}} b_{k,p'}(f)$$

$$\leq K_1 + 16 c_p c_\infty \sum_{n=2}^{\infty} n \cdot 2^{-(n+1)/p} \sum_{k\in\sigma_{n-1}} b_{k,p'}(f).$$

Furthermore there is a constant K_2 such that for all n and $k \in \sigma_{n-1}$,

$$16c_p c_\infty n\, 2^{-(n+1)/p} \sum_{k\in\sigma_{n-1}} b_{k,p'}(f)$$

$$\leq K_2 \sum_{k\in\sigma_{n-1}} k^{-1/p}(\mathrm{Log}\ k)b_{k,p'}(f).$$

From which

$$\pi_{p'}(T'_f) \leq K_1 + K_2 \sum_{k=1}^{\infty} k^{-1/p}(\mathrm{Log}\ k)b_{k,p'}(f).$$

Since by assumption the sum on the right is finite, we have $T'_f \in \Pi_{p'}(L_p, L_\infty)$ which proves the theorem.

One fact concerning functions f which satisfy (*) follows immediately from Corollary 4.3.

<u>Corollary 4.7</u>. If $f \in L_p(0,1)$ satisfies (*) of Theorem 4.6, then $\sum_n \hat{f}(n)$ converges.

Some remaining questions are: Must such an f be continuous? If so, must its fourier series converge uniformly?

<u>Example 4.8</u>. Let $1 < p < \infty$, $p \neq 2$, and $1/q = 1 - |1/2 - 1/p|$. There exists a convolution operator $T_f \in (L_p(0,1),$ defined by a function $f \in L_{p'}(0,1)$ such that $(\hat{f}(n)) \in \ell_q$ but $(\hat{f}(n)) \notin \ell_r$ for any $r < q$. In particular the operator T_f is a diagonal nuclear operator

whose eigenvalues acheive the summability limit q of Theorem 3.5.

By duality, we may assume $1 < p < 2$. Let

$$f(x) = \sum_{n=2}^{\infty} \frac{e^{in \operatorname{Log} n}}{n^{1/2+1/p'} (\operatorname{Log} n)^{\beta}} e^{2\pi inx}, \ \beta > 2.$$

It is shown in Zygmund [32] that f is continuous. Moreover, $1/2 + 1/p' = 3/2 - 1/p = 1/q$, so

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^q = \sum_{n=2}^{\infty} (n^{1/2+1/p'} (\operatorname{Log} n)^{\beta})^{-q}$$

$$= \sum_{n=2}^{\infty} n^{-1} (\operatorname{Log} n)^{-q\beta} < \infty.$$

But for $r < q$, $r(1/2 + 1/p') = r/q < 1$ and $\Sigma|f(n)|^r = \infty$.

We have left to show that $T_f\colon L_p \to L_p$ is nuclear. We show that f satisfies (*) of Theorem 4.5.

For simplicity write $a = 0$ and

$$a_k := k^{-1/2-1/p'} (\operatorname{Log} k)^{-\beta}, \quad k > 1,$$

so that

$$f(x) = \sum_{k=2}^{\infty} a_k \exp\,(ik \operatorname{Log} k + 2\pi ikx).$$

Let

$$p_n(x) := \sum_{k=1}^{n} \exp\,(ik \operatorname{Log} k + 2\pi ikx)$$

By Zygmund ([32], 5.32),

$$|p_n(x)|_\infty = O(n^{\frac{1}{2}}).$$

So summing by parts

$$s_{n+1}(f)(x) = \sum_{k=2}^{n} a_k \exp(ik\mathrm{Log}\, k + 2\pi ikx)$$
$$= p_n(x)a_n + \sum_{k=1}^{n-1} p_k(x)\Delta a_k ,$$

where $\Delta a_k = a_k - a_{k+1}$.

Now since f is continuous,

$$b_{n,p'}(f) = \|t_n(f)\|_{p'} \le \|t_n(f)\|_\infty$$
$$= \lim_{N\to\infty} \|s_{N+1}(f) - s_n(f)\|_\infty,$$

and for $n > 2$ and $N > n$,

$$\|s_{N+1}(f) - s_n(f)\|_\infty$$
$$= \Big\|\sum_{k=1}^{N-1} p_k(x)\Delta a_k + p_N(x)a_N - \sum_{k=1}^{n-2} p_k(x)\Delta a_k - p_{n-1}(x)a_{n-1}\Big\|_\infty$$
$$\le \|p_N(x)\|_\infty a_N + \|p_{n-1}(x)\|_\infty a_{n-1} + \sum_{k=n-1}^{N-1} \|p_k(x)\|_\infty |\Delta a_k|$$
$$\le c_1\Big(N^{\frac{1}{2}}a_N + (n-1)^{\frac{1}{2}}a_{n-1} + \sum_{k=n-1}^{N-1} k^{\frac{1}{2}}|\Delta a_k|\Big)$$

where $c_1 > 0$ is a constant independent of N and n.

By the mean value theorem,

$$\Delta a_k = O(k^{-3/2-1/p'}(\mathrm{Log}\, k)^{-\beta}),$$

so taking limits as $N \to \infty$, there is a $c_2 > 0$ such that

$$b_{n,p'}(f)$$
$$\le c_2\Big((n-1)^{-1/p'}(\mathrm{Log}(n-1))^{-\beta} + \sum_{k=n-1}^{\infty} k^{-1-1/p'}(\mathrm{Log}\, k)^{-\beta}\Big).$$

Finally,

$$\sum_{n=3}^{\infty} n^{-1/p}(\text{Log } n)b_{n,p'}(f) \le c_2 \sum_{n=3}^{\infty} n^{-1}(\text{Log } n)^{1-\beta}$$

$$+ c_2 \sum_{n=3}^{\infty} \sum_{k=n-1}^{\infty} n^{-1/p} k^{-1-1/p'}(\text{Log } k)^{-\beta}\text{Log } n.$$

Denoting the first sum on the right hand side of this inequality by S_1, and the second by S_2, $S_1 < \infty$ since $\beta > 2$. Moreover,

$$S_2 = \sum_{k=2}^{\infty} k^{-1-1/p'}(\text{Log } k)^{-\beta} \sum_{n=3}^{k+1} n^{-1/p}\text{Log } n$$

$$\le \sum_{k=2}^{\infty} k^{-1-1/p'}(\text{Log } k)^{-\beta} \int_1^{k+1} x^{-1/p}\text{Log } x \, dx.$$

Now

$$\int_1^{k+1} x^{-1/p}\text{Log } x \, dx = O(k^{1/p'}\text{Log } k),$$

so for some constant K,

$$S_2 \le K \sum_{k=2}^{\infty} k^{-1}(\text{Log } k)^{1-\beta} < \infty.$$

Hence f satisfies (*) which completes the proof.

Example 4.8 shows that in the extremely important case of convolution operators on L_p-spaces, nuclear operators which are diagonal with respect to a conditional basis may not have absolutely summable eigenvalue (diagonal) sequences, and, in fact, given $q < 2$ there is a nuclear diagonal operator whose diagonal sequence is not in ℓ_q.

We note however that with respect to some conditional bases nuclear diagonal operators must have absolutely summable eigenvalue sequences. A trivial example of this is on Hilbert space (which has conditional bases, see, e.g. [15]) where all eigenvalue sequences of nuclear operators are absolutely summable. For completeness, we present the following less obvious example on the non-reflexive space c_0.

Example 4.9. For $n \in N$, define

$$x_n := (\underbrace{1, \ldots, 1}_{n}, 0, 0, \ldots) \in c_0, \quad \text{and}$$

$$f_n := (\underbrace{0, 0, \ldots, 1}_{n}, -1, 0, \ldots) \in \ell_1.$$

Then (x_n) is a conditional basis for c_0 (see [9]) with associated coefficient functionals (f_n). If $\Sigma \lambda_n f_n \otimes x_n$ is nuclear, then $(\lambda_n) \in \ell_1$.

To see this, note that $T^* \in \mathcal{N}(\ell_1)$, and so by a result of Grothendieck's [5], the image under T^* of the unit ball of ℓ_1 is lattice bounded. Thus if we denote by $\phi\colon c_0 \to \ell_\infty = c_0''$ the canonical injection, there exists a positive sequence $(b_k) \in \ell_1$ such that for all $u \in \ell_1$, $\|u\| \le 1$,

$$|(T^*u)_k| = |(\Sigma \lambda_n (\phi(x_n))(u) f_n)_k| \le b_k, \quad k \in N.$$

(Here $(\cdot)_k$ denotes the k^{th} coordinate).

For any fixed k let $u = \frac{1}{2} f_k$. Then $\|u\| = 1$, and so

$$|(T^*u)_k| = |(\tfrac{1}{2}\ \Sigma\ \lambda_n(\phi(x_n))(f_k)\ f_n)_k|$$

$$= |(\tfrac{1}{2}\ \lambda_k f_k)_k| = |\tfrac{1}{2}\ \lambda_k|.$$

Thus $|\lambda_k| \leq 2\ b_k$ for all k, which means $(\lambda_n) \in \ell_1$.

One final question remains concerning the eigenvalue summability of diagonal nuclear operators: Is there a space X with conditional basis (x_n) and a diagonal operator $T \in \mathcal{N}(X)$ defined by a sequence (λ_n) such that $(\lambda_n) \in \ell_2$ but $(\lambda_n) \notin \ell_p$ for any $p < 2$?

CHAPTER 5

NUCLEAR OPERATORS OF DIAGONAL TYPE

Up to this point we have constructed nuclear operators whose eigenvalue sequences exhibit desired summability properties. These examples provide little information regarding the more general question originally raised by Pełczyński and Saphar: Given an arbitrary sequence $(\lambda_n) \in \ell_2$, is there a nuclear operator on some Banach space for which the nonzero elements of (λ_n) are an eigenvalue sequence? This question obviously requires consideration of a larger class of nuclear operators than those which are diagonal relative to a basis since such operators must have eigenvalue sequences which are at least conditionally convergent by Corollary 4.3.

In fact, until recently there was no example of a nuclear operator whose eigenvalue sequence was not conditionally convergent. H. König, in an unpublished example, constructed a nuclear operator on c_0 with eigenvalue sequence $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots$. Briefly, he used weighted tails of Davie's matrix (see, e.g., Lindenstrauss and Tzafriri 15 , p. 87) as blocks, each having the single eigenvalue 2^{-n} with multiplicity 2^n. The direct sum of these blocks, each viewed as an operator on a copy of c_0, is then shown to be nuclear. Since the resulting operator is not of diagonal type, and, more importantly, since we will be able to reproduce this eigenvalue sequence as a specific case of a general example in Chapter 6, we omit the details of König's construction.

In the remark following Theorem 3.3 it was shown that every sequence $(\lambda_n) \in \ell_2$ is a subsequence of the eigenvalue sequence of a nuclear operator. One major defect of that construction, however, is that the remaining eigenvalues, if any, are unknown. We will now introduce a class of operators of diagonal type with which we can improve that result in the sense that these excess eigenvalues are known and bear a simple relationship to the given sequence. Then, in Chapter 6, we will discuss the Pełczyński-Saphar problem in complete generality.

If X is a Banach space with a basis (x_n), we can identify X with a space of sequences, or <u>sequence space</u>, by the correspondence

$$x = \Sigma\, a_n x_n \leftrightarrow (a_n).$$

<u>Definition 5.1</u>. Let $X_1, \ldots, X_n$ be sequence spaces and $D_i\colon X_i \to X_{i+1}$ $(D_n\colon X_n \to X_1)$ be diagonal operators defined by sequences $(\mu_{ij})_{j=1}^{\infty}$ for $i = 1, \ldots, n$. Then

$$D = \bigoplus_{i=1}^{n} D_i\colon \left(\bigoplus_{i=1}^{n} X_i\right)_\infty \to \left(\bigoplus_{i=1}^{n} X_i\right)_\infty$$

is called a <u>cyclic diagonal operator</u> on the space $\left(\bigoplus_{i=1}^{n} X_i\right)_\infty$.

The eigenvalue sequences of cyclic diagonal operators are related to the collection of diagonal sequences in the following way.

Proposition 5.2. Let $D: \bigoplus_{i=1}^{n} X_i \to \bigoplus_{i=1}^{n} X_i$ be a cyclic diagonal operator such that each component diagonal sequence (μ_{ij}) is entirely nonzero. Then the eigenvalues of D are exactly all of the n^{th} roots of the numbers

$$\beta_j := \prod_{i=1}^{n} \mu_{ij}, \quad j = 1,2,\ldots .$$

Proof. Suppose $\lambda \neq 0$ is an eigenvalue of D and

$$((\alpha_{1j}),(\alpha_{2j}),\ldots ,(\alpha_{nj})) \neq 0$$

is a corresponding eigenvector. For some i_0, j_0, we have $\alpha_{i_0 j_0} \neq 0$. Let D^{j_0} be the operator D restricted to only the $j_0{}^{th}$ coordinates of $\oplus X_i$. Then

$$\begin{aligned}\lambda\,(\alpha_{1j_0},\alpha_{2j_0},\ldots ,\alpha_{nj_0}) &= D^{j_0}(\alpha_{1j_0},\alpha_{2j_0},\ldots ,\alpha_{nj_0})\\ &= (\mu_{nj_0}\alpha_{nj_0},\mu_{1j_0}\alpha_{1j_0},\ldots ,\mu_{(n-1)j_0}\alpha_{(n-1)j_0})\end{aligned}$$

and we have the following n equations:

$$\begin{aligned}\lambda\,\alpha_{1j_0} &= \mu_{nj_0}\alpha_{nj_0}\\ \lambda\,\alpha_{2j_0} &= \mu_{1j_0}\alpha_{1j_0}\\ &\cdots\\ \lambda\,\alpha_{nj_0} &= \mu_{(n-1)j_0}\alpha_{(n-1)j_0} .\end{aligned}$$

Since λ, $\alpha_{i_0 j_0}$, and all of the μ_{ij_0}'s are nonzero, the cyclic nature of the equations guarantees that $\alpha_{ij_0} \neq 0$ for all i. The product of these equations is

$$\lambda^n \prod_{i=1}^{n} \alpha_{ij_0} = \prod_{i=1}^{n} \mu_{ij_0}\alpha_{ij_0} \ ,$$

so

$$\lambda^n = \prod_{i=1}^{n} \mu_{ij_0} = \beta_{j_0}.$$

Conversely, if ω is an arbitrary n^{th} root of β_{j_0}, we can produce an eigenvector $((\alpha_{1j}),(\alpha_{2j}),\ldots,(\alpha_{nj}))$ with eigenvalue ω by setting $\alpha_{ij} = 0$ for $j \neq j_0$, $\alpha_{1j_0} = 1$, and solving recursively:

$$\alpha_{2j_0} = \mu_{1j_0}/\omega \ , \quad \alpha_{3j_0} = \mu_{1j_0}\mu_{2j_0}/\omega^2, \quad \text{etc.}$$

So each n^{th} root is an eigenvalue of D.

<u>Corollary 5.3</u>. Suppose $D\colon \bigoplus_{i=1}^{n} X_i \to \bigoplus_{i=1}^{n} X_i$ is a cyclic diagonal operator defined by sequences $(\mu_{ij}) \in \ell_{p_i}$, where $0 < p_i \leq \infty$. Then

$$(\lambda_n(D)) \in \ell_r \qquad \text{where} \qquad \frac{1}{r} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{p_i}.$$

<u>Proof</u>. Consider the sequence $(|\lambda_n(D)|)$. By Theorem 5.2 this sequence consists of the numbers $|\beta_j|^{1/n}$, $j = 1,2,\ldots,$ each repeated n times. So it suffices to show that

$$\sum_{j=1} \left|\prod_{i=1}^{n} \mu_{ij}\right|^{r/n} < \infty \quad \text{where} \quad \left(\frac{r}{n}\right)^{-1} = \sum_{i=1}^{n} \frac{1}{p_i} \ .$$

To simplify notation, let $\frac{1}{s_k} = \sum_{i=1}^{k} \frac{1}{p_i}$. Then $s_n = r/n$

and $1/s_{k-1} + 1/p_k = 1/s_k$.

We proceed by induction. The case $n = 1$ is trivial. If

$$\sum_{j=1}^{\infty} \left|\prod_{i=1}^{n-1} \mu_{ij}\right|^{s_{n-1}} < \infty,$$

then

$$\left(\left|\prod_{i=1}^{n-1} \mu_{ij}\right|^{s_n}\right) \in \ell_{s_{n-1}/s_n} .$$

But $(|\mu_{nj}|^{s_n}) \in \ell_{p_n/s_n}$, and

$$\begin{aligned}(s_{n-1}/s_n)^{-1} + (p_n/s_n)^{-1} &= s_n(1/s_{n-1} + 1/p_n) \\ &= 1.\end{aligned}$$

So s_{n-1}/s_n and p_n/s_n are conjugate exponents. Thus

$$\sum_{j=1}^{\infty} \left|\prod_{i=1}^{n-1} \mu_{ij}\right|^{s_n} |\mu_{nj}|^{s_n} = \sum_{j=1}^{\infty} \left|\prod_{i=1}^{n} \mu_{ij}\right|^{r/n} < \infty.$$

The exponent r in Corollary 5.3 "averages" the the exponents $p_1, \ldots, p_n$. For this reason we will call r the <u>summability</u> <u>mean</u> of the exponents $p_1, \ldots, p_n$.

A cyclic diagonal operator D is nuclear if and only if each component mapping D_i is nuclear. Thus we will need some criteria for nuclearity of diagonal operators between sequence spaces. The following is a well-known result (see Tong [30] and Holub [8]) concerning diagonal operators between ℓ_p spaces.

<u>Note</u>. Henceforth, if $p = \infty$, by ℓ_p we will mean c_0.

<u>Theorem 5.4</u>. Suppose $1 \leq p,q \leq \infty$ and $D: \ell_p \to \ell_q$ is a diagonal operator defined by a sequence (μ_n). Then D is nuclear if and only if $(\mu_n) \in \ell_r$, where

(a) $r = 1$ for $p \geq q$,

(b) $1/r = 1 - (1/p - 1/q)$ for $p < q$.

Combining Corollary 5.3 and Theorem 5.4 we can show that nuclear cyclic diagonal operators need not have absolutely summable eigenvalue sequences.

<u>Proposition 5.5</u>. Let $X = \bigoplus_{i=1}^{n} \ell_{p_i}$, where $1 \leq p_i \leq \infty$ for $i = 1,\ldots,n$, and assume that not all of the p_i's are equal. Then there is a cyclic diagonal operator $D \in \mathcal{N}(X)$ such that $(\lambda_j(D)) \notin \ell_1$.

<u>Proof</u>. Since not all of the p_i's are equal, either there is an $i < n$ such that $p_i < p_{i+1}$ or $p_n < p_1$. We will assume (rearranging if necessary) that $p_n < p_1$. Let

$$1/s := 1 - (1/p_n - 1/p_1), \quad \text{and}$$

$$1/r := 1/n\,(n - 1 + 1/s).$$

Then $s > 1$, and so $r > 1$.

Let $(\lambda_j) \in \ell_r$ be such that $(\lambda_j) \notin \ell_u$ for any $u < r$, and define the cyclic diagonal operator D on X by the diagonal sequences

$$(\mu_{ij}) = (\lambda_j^{\,r}) \in \ell_1 \quad \text{for} \quad i = 1,\ldots,n-1$$

$$(\mu_{nj}) = (\lambda_j^{\,r/s}) \in \ell_s\ .$$

Then each component operator D_i is nuclear by Theorem 5.4, so D is nuclear. Moreover, r is the summability mean of the summability exponents of the sequences (μ_{ij}), $i = 1,\ldots,n$, so by Corollary 5.3, $(\lambda_j(D)) \in \ell_r$. But for each j,

$$\prod_{i=1}^{n} \mu_{ij} = \lambda_j^{r(n-1+1/s)} = \lambda_j^{n},$$

and thus by Proposition 5.2 (λ_j) is a subsequence of $(\lambda_j(D))$. By assumption, $(\lambda_j) \notin \ell_1$, so neither is $(\lambda_j(D))$.

The case in which D is a cyclic diagonal operator on the direct sum of only two ℓ_p spaces is of particular interest.

Proposition 5.6. Suppose $1 \leq p \leq q \leq \infty$.

(a) Define $D\colon \ell_q \oplus \ell_p \to \ell_q \oplus \ell_p$ by

$$D((a_n),(b_n)) := ((\beta_n b_n),(\alpha_n a_n)).$$

Then the eigenvalues of D are $\pm\sqrt{\alpha_n\beta_n}$, $n=1,2,\ldots$ Consequently, if D is nuclear, then $(\lambda_n(D)) \in \ell_s$, where $1/s = 1 - \frac{1}{2}(1/p - 1/q)$.

(b) Conversely, if (λ_n) is a nonzero sequence in ℓ_s $(1 \leq s \leq 2)$, and p and q satisfy $1/s = 1 - \frac{1}{2}(1/p - 1/q)$, then there is a nuclear cyclic diagonal operator $D\colon \ell_q \oplus \ell_p \to \ell_q \oplus \ell_p$ with eigenvalue sequence $(\lambda_1,-\lambda_1,\lambda_2,-\lambda_2,\ldots)$.

<u>Proof</u>. (a) The operator D is a cyclic diagonal operator defined by the sequences (α_n) and (β_n). By Proposition 5.2, the eigenvalues of D are as indicated. If D is nuclear, then $(\alpha_n) \in \ell_1$ and $(\beta_n) \in \ell_r$, where $1/r = 1 - (1/p - 1/q)$, by Theorem 5.4. So by Corollary 5.3, $(\lambda_n(D)) \in \ell_s$, where $1/s = \frac{1}{2}(1 + 1 - (1/p - 1/q)) = 1 - \frac{1}{2}(1/p - 1/q)$.

(b) First assume that p and q can be chosen satisfying $1/s = 1 - \frac{1}{2}(1/p - 1/q)$ with either $q < \infty$ or $p > 1$. Then $1/p - 1/q < 1$ so $s < 2$. Set $(\alpha_n) = (\lambda_n{}^s) \in \ell_1$ and $(\beta_n) = (\lambda_n{}^{2-s}) \in \ell_r$, where

$$\begin{aligned} 1/r = (2 - s)/s &= 2/s - 1 \\ &= 2(1 - \tfrac{1}{2}(1/p - 1/q)) - 1 \\ &= 1 - (1/p - 1/q). \end{aligned}$$

If we define D as in part (a) then its eigenvalues are $\pm\sqrt{\alpha_n\beta_n} = \pm\lambda_n$ and its component mappings are nuclear by 5.4. But then D is nuclear and has the required eigenvalue sequence.

There remains the case in which p and q are chosen with $q = \infty$ and $p = 1$. This means $s = 2$, i.e., $(\lambda_n) \in \ell_2$. Since $(\lambda_n{}^2) \in \ell_1$, by Lemma 3.2 there are sequences $(\alpha_n) \in \ell_1$ and $(\beta_n) \in c_0$ such that $\alpha_n\beta_n = \lambda_n{}^2$ for all n. Proceeding as above, the cyclic diagonal operator on $c_0 \oplus \ell_1$ defined as in part (a) has all of the required properties.

<u>Remark</u>. Since for $p \leq 2 \leq q$, $\ell_q \oplus \ell_p$ is of type p and cotype q, it is interesting to relate part (b) of 5.6 to the known and conjectured bounds on summability exponents of nuclear operators on spaces of given type and cotype. If we denote by r_k the known bound on summability exponents given in Theorem 3.8, r_c the (smaller) conjectured bound mentioned in the remarks following that theorem, and r_D the exponent that can be attained in 5.6(b), then

$$1/r_k = 1 - (1/p - 1/q) = 1 - ((1/p - 1/2) + (1/2 - 1/q))$$

$$1/r_c = 1 - \max\,(1/p - 1/2,\ 1/2 - 1/q), \quad \text{and}$$

$$1/r_D = 1 - \tfrac{1}{2}(1/p - 1/q) = 1 - \text{average}(1/p - 1/2, 1/2 - 1/q)$$

So $r_D \leq r_c \leq r_k$. If p or $q = 2$, then $r_c = r_k$ and this bound is attained using the convolution operators on $L_p(0,1)$ of Example 4.8. Aside from the trivial case $p = q = 2$, the only case in which $r_D = r_c$ is when $q = p'$. So only in the "balanced" case $\ell_{p'} \oplus \ell_p$ can we find a nuclear cyclic diagonal operator which has an eigenvalue sequence whose minimum exponent of summability is r_D, the conjectured limit.

If $p = 1$ and $q = \infty$ we have the extreme case ($s = 2$ in Proposition 5.6(b)). This provides an important example in the study of nuclear operators of diagonal type. (recall that Example 4.8 only gave summability exponents strictly less than 2).

<u>Example 5.7</u>. Every nonzero sequence $(\lambda_n) \varepsilon \ell_2$ is a subsequence of the eigenvalue sequence of a nuclear operator D of diagonal type. Moreover, D can be chosen so that $(\lambda_n(D))$ consists only of the sequences (λ_n) and $(-\lambda_n)$.

By 5.6(b) there is a nuclear cyclic diagonal operator $D\colon (c_0 \oplus \ell_1)_\infty \to (c_0 \oplus \ell_1)_\infty$ which has the required eigenvalue sequence. So we merely must show that this operator D is of diagonal type, i.e., that it has a diagonal representation in terms of a biorthogonal system.

Now D was defined by

$$D((a_j),(b_j)) = ((\beta_j b_j),(\alpha_j a_j))$$

where $(\alpha_j) \varepsilon \ell_1$, $(\beta_j) \varepsilon c_0$, and $\alpha_j\beta_j = \lambda_j^2$ for all j. In the proof of Proposition 5.2 we calculated that an eigenvector corresponding to λ_n (resp. $-\lambda_n$) was $(e_n,\omega_n f_n)$ (resp. $(e_n,-\omega_n f_n)$) where (e_n) and (f_n) are the unit vector bases of c_0 and ℓ_1 and

$$\omega_n = \alpha_n/\lambda_n.$$

So we have a sequence of eigenvectors $((e_n,\pm\omega_n f_n)) \subset (c_0 \oplus \ell_1)_\infty$. We have only to produce a sequence of functionals in $((c_0 \oplus \ell_1)_\infty)' = (\ell_1 \oplus \ell_\infty)_1$ which is biorthogonal to this sequence. But the sequence required is merely

$$((\tfrac{1}{2}f_n,\frac{1}{2\omega_n}g_n)) \subset (\ell_1 \oplus \ell_\infty)_1 \text{ where } g_n=(\delta_{jn})_{j=1}^{\infty} \varepsilon\ \ell_\infty.$$

(The functional action being coordinatewise multiplication followed by summing).

A natural question arising from Example 5.7 is whether the excess "negative" eigenvalues can be somehow eliminated to leave a nuclear operator with eigenvalue sequence exactly (λ_n). The natural method of attempting to do this is to follow the nuclear operator D with a projection onto $[(e_n, \omega_n f_n)]$. We show that in general this cannot be done.

Proposition 5.8. Let $(\lambda_n) \in \ell_s$, $s > 1$, and

$$D: \ell_q \oplus \ell_p \to \ell_q \oplus \ell_p$$

be as in Proposition 5.6(b). The eigenvectors (x_n) of D corresponding to the sequence (λ_n) form an unconditional basic sequence in $\ell_q \oplus \ell_p$. Consequently if $(\lambda_n) \notin \ell_1$, $[x_n]$ is not complemented in $\ell_q \oplus \ell_p$.

Proof. As in Example 5.7 we may write $x_n = (e_n, \omega_n f_n)$, $n = 1,2,\ldots$, where ω_n depends on the defining sequences for D. Let (a_n) be a sequence of scalars and $M < N$. Then

$$\begin{aligned}
\left\| \sum_{n=1}^{M} a_n x_n \right\| &= \left\| \sum_{n=1}^{M} a_n (e_n, \omega_n f_n) \right\| \\
&= \sup\Big(\sum_{n=1}^{M} |a_n|, \sup_{1 \le n \le M} |a_n \omega_n| \Big) \\
&\le \sup\Big(\sum_{n=1}^{N} |a_n|, \sup_{1 \le n \le N} |a_n \omega_n| \Big) \\
&= \left\| \sum_{n=1}^{N} a_n x_n \right\|
\end{aligned}$$

So by Proposition 1.1 (x_n) is a basic sequence.

Similarly, if (b_n) is a scalar sequence with $|b_n| \leq |a_n|$ for all n, then

$$\|\sum_{n=1}^{\infty} b_n x_n\| \leq \|\sum_{n=1}^{\infty} a_n x_n\|.$$

So (x_n) is unconditional.

Finally, if $[x_n]$ is complemented in $\ell_q \oplus \ell_p$, $D|_{[x_n]}$ is a diagonal nuclear operator with respect to (x_n). Hence by Proposition 4.4, $(\lambda_n) \in \ell_1$.

Notice that Proposition 5.8 also applies to the eigenvectors $(\bar{x}_n) = ((e_n, -\omega_n f_n))$ of the subsequence $(-\lambda_n)$ of eigenvalues of D, so the subspaces $[x_n]$ and $[\bar{x}_n]$ have the following properties.

Proposition 5.9. If $D \in \mathcal{N}(\ell_q \oplus \ell_p)$, $(\lambda_n) \notin \ell_1$, (x_n), and $(\bar{x}_n)$ are all as defined above, then $[x_n]$ and $[\bar{x}_n]$ are isometric subspaces of $\ell_q \oplus \ell_p$ which are quasicomplements, i.e., $[x_n] \cap [\bar{x}_n] = \{0\}$, $[x_n] + [\bar{x}_n]$ is dense in $\ell_q \oplus \ell_p$, but $[x_n]$ and $[\bar{x}_n]$ are not complements.

Proof. The mapping

$$\sum_{n=1}^{\infty} a_n x_n \rightarrow \sum_{n=1}^{\infty} a_n \bar{x}_n$$

is an isometry since the norm of each sum is

$$\sup\,(\sum_{n=1}^{\infty} |a_n|, \sup_n |a_n \omega_n|).$$

By the uniqueness of representation in $\ell_q \oplus \ell_p$, $[x_n] \cap [\bar{x}_n] = \{0\}$, and since for all n, $(e_n, 0) = \tfrac{1}{2}(x_n + \bar{x}_n)$

and $(0,f_n) = \frac{1}{2\omega_n}(x_n-\overline{x}_n)$, $[x_n] + [\overline{x}_n]$ is dense in $\ell_q \oplus \ell_p$. Since by Proposition 5.8, neither subspace is complemented, $[x_n]$ and $[\overline{x}_n]$ are quasicomplements.

Let us recapitulate. For a nonzreo sequence $(\lambda_n) \in \ell_s \setminus \ell_1$ where $1 < s \leq 2$, we can define a cyclic diagonal operator $D \in \mathcal{N}(\ell_q \oplus \ell_p)$ where $p < q$ and $1/s = 1 - \frac{1}{2}(1/p - 1/q)$, by

$$D((a_n),(b_n)) = ((\beta_n b_n),(\alpha_n a_n)).$$

The eigenvalue sequence for D is $(\lambda_1,-\lambda_1,\lambda_2,-\lambda_2,\dots)$ and the corresponding eigenvector sequence is

$$(x_1,\overline{x}_1,x_2,\overline{x}_2,\dots) =$$

$$(e_1+\omega_1 f_1, e_1-\omega_1 f_1, e_2+\omega_2 f_2, e_2-\omega_2 f_2,\dots)$$

where (e_n) and (f_n) are the unit bases of ℓ_q and ℓ_p respectively and $\omega_n = \alpha_n/\lambda_n$ for all n. The subspaces $[x_n]$ and $[\overline{x}_n]$ of $\ell_q \oplus \ell_p$ are then isometric quasicomplements in $\ell_q \oplus \ell_p$.

It turns out that spaces of the form $[e_n+\omega_n f_n]$ where (ω_n) is a scalar sequence are a special type of modular sequence space (cf. Lindenstrauss and Tzafriri [15]) which has been extensively studied and has important applications. In particular, suppose $1 \leq p < q < \infty$, $1/r = 1/p - 1/q$, and $\omega = (\omega_n)$ is a sequence of scalars such that

$$(*) \qquad \sum_{n=1}^{\infty} |\omega_n|^r = \infty \quad \text{and} \quad \omega_n \to 0.$$

Then the space

$$X_{q,p,\omega} = [e_n + \omega_n f_n] \subset \ell_q \oplus \ell_p$$

is called a *Rosenthal weighted space*.

We summarize some of the major properties of such spaces (for proofs see, e.g., Lindenstrauss and Tzafriri, [15] and [16]).

(1) If (ω_n) and (ω_n') both satisfy $(*)$, then $X_{q,p,\omega}$ and $X_{q,p,\omega'}$ are isomorphic.

(2) $X_{q,p,\omega} \oplus X_{q,p,\omega}$ is isomorphic to $X_{q,p,\omega}$.

(3) $X_{q,2,\omega}$ is a complemented subspace of $L_q(0,1)$.

Recall from the proof of Proposition 5.6(b) that if $1 \leq p < q < \infty$ (and hence $s < 2$) then the defining sequences (α_n) and (β_n) of the operator D are given by

$$\alpha_n = \lambda_n{}^s, \qquad \beta_n = \lambda_n{}^{2-s}, \qquad\qquad n \in \mathbb{N}.$$

Proposition 5.10. If $1 \leq p < q < \infty$ then the eigenvector subspaces $[x_n]$ and $[\bar{x}_n]$ are Rosenthal weighted spaces.

Proof. Since $[x_n] = [e_n + \omega_n f_n] \subset \ell_q \oplus \ell_p$, we have only to show that (ω_n) satisfies $(*)$.

Now for each n,

$$\omega_n = \alpha_n/\lambda_n = \lambda_n{}^s/\lambda_n = \lambda_n{}^{s-1}.$$

Moreover,

$$1/s = 1 - \tfrac{1}{2}(1/p - 1/q) = 1 - 1/2r,$$

so $|\omega_n|^r = |\lambda_n|^{s/2}$. Since $1 < s < 2$ and $(\lambda_n) \in \ell_s \setminus \ell_1$ we have

$$\sum_{n=1}^{\infty} |\omega_n|^r = \sum_{n=1}^{\infty} |\lambda_n|^{s/2} = \infty, \quad \text{and} \quad \omega_n = \lambda_n^{s-1} \to 0.$$

Of interest here is the fact that we can generate a space $[x_n]$ for every pair of exponents (p,q) with $1 \leq q < p < \infty$. By property (1) of a Rosenthal weighted space, this means that essentially "all" of the spaces $X_{q,p,\omega}$ arise naturally as eigenvector spans of certain nuclear cyclic diagonal operators. In addition, the fact that these subspaces are not complemented (and indeed can be matched in quasicomplemented pairs) is a direct consequence of this method of derivation.

CHAPTER 6

THE PEŁCZYŃSKI-SAPHAR PROBLEM

In this final chapter we settle the long outstanding problem of Pełczyński and Saphar mentioned previously: Which (nonzero) sequences in ℓ_2 are eigenvalue sequences of nuclear operators on some Banach space? Our procedure consists of proving a result concerning absolutely 2-summing operators which is valid for arbitrary Banach spaces and then applying this result to a remarkable space recently constructed by G. Pisier [24]. In particular, we are able to use operators of diagonal type to answer this question.

We state first the properties of Pisier's space which we need.

<u>Theorem 6.1</u>. (Pisier, 1981). There exists an infinite dimensional separable Banach space P such that

(1) $P \hat{\otimes}_\varepsilon P \simeq P \hat{\otimes}_\pi P$,

(2) P and P' have cotype 2, and

(3) Every operator on P which is the norm limit of finite rank operators is nuclear.

The symbols $\hat{\otimes}_\varepsilon$ and $\hat{\otimes}_\pi$ refer to the injective and projective tensor products, respectively. (for definitions see, e.g., [28]). We will not attempt to reproduce a proof of 6.1 since it requires a great deal of machinery beyond the scope of this paper. We would point out, however, that property (3), along with many other properties of P (it does not have the approximation property, each of its finite dimensional subspaces has "worst possible" projection constant, etc.), is a consequence of the first two properties. We have emphasized property (3) since it is the property of P which applies directly to our study. By (3), in order to find a nuclear operator on P with a given eigenvalue sequence, we have only to construct an operator in $\overline{\mathcal{F}}(P)$ with that sequence. It happens that if we "expand" our search to $\overline{\mathcal{F}}$, then every nonzero sequence $(\lambda_n) \in \ell_2$ can be realized as an eigenvalue sequence.

<u>Theorem 6.2</u>. Let X be an infinite dimensional Banach space and $(\lambda_n) \in \ell_2$ with $\lambda_n \neq 0$, $n = 1,2,\ldots$. There exists an operator of diagonal type $T \in \overline{\mathcal{F}}(X) \cap \Pi_2(X)$ with eigenvalue sequence (λ_n).

<u>Proof</u>. We will use the following result concerning basic sequences in Banach spaces.

<u>Lemma 6.3</u>. Let X be an infinite dimensional Banach space and $(\alpha_n) \in c_0$ with $1 > \alpha_1 \geq \alpha_2 \geq \ldots > 0$. Then there is a basic sequence $(x_n) \subset X$ and functionals $(f_n) \subset X'$ such that

(a) $\sup_{\|f\|=1} \{ \sum_{n=1}^{\infty} |f(x_n)|^2 \} < \infty$

(b) $f_n(x_k) = \alpha_n \delta_{nk}$, and

(c) $\|f_n\| \to 0$.

Proof of Lemma. We will use the following notation. For a finite sequence $x_1, \ldots, x_n$ in a Banach space Y, let

$$\Theta(x_i) = \min_{1 \le k < n} \{ I([x_i : i \le k]; [x_i : i \le n]) \}$$

where for any finite dimensional subspaces Y_1 and Y_2 of Y,

$$I(Y_1; Y_2) = \inf \{ \|y_1 - y_2\| : y_1 \in Y_1, \|y_1\| = 1, y_2 \in Y_2 \}.$$

The following fact is well-known (see [18]).

(*) If Y is infinite dimensional and $\varepsilon > 0$, then for any $n \in N$ there is a sequence $y_1, \ldots, y_n$ such that $\|y_i\| = 1$, $\Theta(y_i) \ge 1/(1 + \varepsilon)$ and

$$\sup_{\|f\|=1} \{ (\sum_{i=1}^{n} |f(y_i)|^2)^{\frac{1}{2}} \le 1 + \varepsilon.$$

So let $\varepsilon > 0$, $\beta_n = \alpha_n^{\frac{1}{2}}$ and (n_k) be an increasing sequence of integers with $n_0 = 0$ such that $\beta_n \le 2^{-k}$ for $n > n_k$. Let $\sigma_k = \{n_{k-1}+1, \ldots, n_k\}$. Choose $(y_i)_{i \varepsilon \sigma_1}$ as in (*). Let $x_i = \beta_i y_i$, $i \in \sigma_1$. Then $\|x_i\| = \beta_i$, $\Theta(x_i) \ge 1/(1 + \varepsilon)$, and

$$\sup_{\|f\|=1} \{ (\sum_{i \varepsilon \sigma_1} |f(x_i)|^2)^{\frac{1}{2}} \} \le (\sup_{i \varepsilon \sigma_1} \beta_i)(1 + \varepsilon) \le 1 + \varepsilon.$$

Guararii [7] has shown that if Y is a finite dimensional subspace of an infinite dimensional space X, then there is an infinite dimensional subspace Z of X such that $I(Y;Z) > 1 - \varepsilon$. So if $Y_1 = [x_i : i\varepsilon\sigma_1]$, let Z_1 be an infinite dimensional subspace of X with $I(Y_1,Z_1) > 1-\varepsilon$. Again by (*) we can choose $(y_i)_{i\varepsilon\sigma_2} \subset Z_1$ so that if $x_i = \beta_i y_i$, $i \varepsilon \sigma_2$, we have $\|x_i\| = \beta_i$, $\theta(x_i) \geq 1/(1 + \varepsilon)$, and

$$\sup_{\|f\|=1} \{(\sum_{i\varepsilon\sigma_2} |f(x_i)|^2)^{\frac{1}{2}}\} \leq (\sup_{i\varepsilon\sigma_2} \beta_i)(1 + \varepsilon) = 2^{-1}(1+\varepsilon).$$

Let $Y_2 = [x_i : i\varepsilon\sigma_2]$ and let Z_2 be an infinite dimensional subspace of X with $I(Y_1 \oplus Y_2 ; Z_2) > 1 - \varepsilon$. Proceeding as before, we obtain a sequence $(x_i)_{i\varepsilon\sigma_3} \subset Z_2$ with $\|x_i\| = \beta_i$, $\theta(x_i) \geq 1/(1 + \varepsilon)$, and

$$\sup_{\|f\|=1} \{(\sum_{i\varepsilon\sigma_3} |f(x_i)|^2)^{\frac{1}{2}}\} \leq 2^{-2}(1 + \varepsilon).$$

In this manner we generate a sequence $(x_n)_{n=1}^{\infty}$ with $\|x_n\| = \beta_n$ for all n, and

$$\sup_{\|f\|=1} \{(\sum_{n=1}^{\infty} |f(x_n)|^2)^{\frac{1}{2}}\} \leq \sum_{k=1}^{\infty} \sup_{\|f\|=1} \{(\sum_{i\varepsilon\sigma_k} |f(x_i)|^2)^{\frac{1}{2}}\}$$

$$\sum_{k=1}^{\infty} 2^{-k}(1 + \varepsilon) < \infty.$$

Furthermore, for all n, $\theta(x_i)_{i\varepsilon\sigma_n} \geq 1/(1 + \varepsilon)$ and

$$I(Y_1 \oplus \dots \oplus Y_n ; Y_{n+1}) > 1 - \varepsilon,$$

so by another result of Guararii [7], (x_n) is a basic sequence.

But now we may define a sequence of functionals (f_n) on the span of the x_n's with norm $\|f_n\| = \beta_n$ by $f_n(x_k) = \beta_n^2\delta_{nk} = \alpha_n\delta_{nk}$. Since $\beta_n \to 0$, extending these functionals to all of X by the Hahn-Banach Theorem completes the proof.

Using this lemma we prove the theorem as follows.

Given (λ_n), since $(\lambda_n^2) \in \ell_1$ we can use Lemma 3.2 to find sequences $(\alpha_n) \in c_0$ and $(\beta_n) \in \ell_2$ such that (α_n) satisfies the condition of Lemma 6.3 and $\alpha_n\beta_n = \lambda_n$. We thus can choose a basic sequence (x_n) and a sequence of functionals (f_n) as in that lemma and define the following operators.

$A: X \to c_0$ defined by $Ax = (f_n(x))$,

$B: c_0 \to \ell_2$ defined by $B(\xi_n) = (\beta_n\xi_n)$, and

$C: \ell_2 \to X$ defined by $C(\xi_n) = \Sigma\xi_n x_n$.

Now $\|Ax\| \leq \|x\| \max\limits_n \|f_n\|$, so $A \in \mathcal{L}(X,c_0)$. Since $(\beta_n) \in \ell_2$, $B \in \mathcal{L}(c_0,\ell_2)$, and finally

$$\|C(\xi_n)\| = \|\sum_{n=1}^{\infty} \xi_n x_n\| = \sup_{\|f\|=1} |\sum_{n=1}^{\infty} \xi_n f(x_n)|$$

$$\leq (\sum_{n=1}^{\infty} |\xi_n|^2)^{\frac{1}{2}} \sup_{\|f\|=1} (\sum_{n=1}^{\infty} |f(x_n)|^2)^{\frac{1}{2}}$$

so by 6.3(a), $C \in \mathcal{L}(\ell_2,X)$.

Let $T = CBA$. Then

$$Tx = CBAx = CB(f_n(x)) = C(\beta_n f_n(x)) = \sum_{n=1}^{\infty} \beta_n f_n(x)x_n.$$

But (x_n) is a basic sequence with coefficient functionals $((1/\alpha_n)f_n)$, i.e., T has diagonal representation

$$T = \sum_{n=1}^{\infty} \lambda_n((1/\alpha_n)f_n) \otimes x_n.$$

So the eigenvalues of T are (λ_n) by 4.1.

Since $\Pi_2(c_0, \ell_2) = \mathcal{L}(c_0, \ell_2)$, $T \in \Pi_2(X)$ by the ideal property. Finally, to prove that $T \in \overline{\mathcal{F}}(X)$, it suffices to show that $A \in \overline{\mathcal{F}}(X, c_0)$, i.e., that

$$\lim_{N\to\infty} \|(f_n(x))_{n=N}^{\infty}\|_\infty = 0 \text{ uniformly for } \|x\| \leq 1.$$

Now

$$\begin{aligned} \|(f_n(x))_{n=N}^{\infty}\|_\infty &= \sup_{n\geq N} |f_n(x)| \\ &\leq \|x\| \sup_{n\geq N} \|f_n\| \\ &\leq \sup_{n\geq N} \|f_n\|. \end{aligned}$$

By Lemma 6.3, $\|f_n\| \to 0$, so the limit is uniform and $T \in \overline{\mathcal{F}}(X)$.

Combining 6.1 and 6.2 we can state our final example.

<u>Example 6.4</u>. Let $(\lambda_n) \in \ell_2$, $\lambda_n \neq 0$ for all n. Then there is a nuclear operator of diagonal type on Pisier's space which has eigenvalue sequence exactly (λ_n).

As a final remark we note that once we have constructed a nuclear operator T having a given eigenvalue sequence on Pisier's space, then by Proposition 3.1 we can factor T as follows:

$$T: P \xrightarrow{A} c_0 \xrightarrow{D} \ell_1 \xrightarrow{B} P$$

where $D \in \mathcal{N}(c_0, \ell_1)$. Then the related operators $ABD \in \mathcal{N}(c_0)$ and $DAB \in \mathcal{N}(\ell_1)$ have the same eigenvalue sequence as T. Thus the Pelczynski-Saphar question can also be answered affirmatively using nuclear operators on either c_0 or ℓ_1.

REFERENCES

1. N. Dunford and J. Schwartz, Linear Operators I, Interscience, New York, 1958.

2. Per Enflo, "A counterexample to the approximation problem in Banach spaces," Acta Math. 13(1973) pp. 261-266.

3. J. Gills, Jr., Totally Factorable Operators, Dissertation, Louisiana State University, 1977.

4. A. Grothendieck, "Sur une notion de produit tensoriel topologique d'espaces vectoriels topologiques, et une classe remarquable d'espaces vectoriels liées a cette notion," C. R. Acad. Sci. Paris 233(1951) pp. 1556-1558.

5. A. Grothendieck, Produits Tensoriels Topologiques et Espaces Nucléaires, Mem. Amer. Math. Soc. 16, Providence, 1955.

6. A. Grothendieck, "Résumé de la théorie métrique des produits tensoriels topologiques," Bol. Soc. Math. Sao Paolo 8(1956) pp. 1-79.

7. V. I. Guararii, "Inclinations of subspaces and conditional bases in Banach spaces," Dokl. Akad. Nauk SSSR 145(1962) pp. 504-506.

8. J. R. Holub, "Diagonal nuclear maps in sequence spaces," Math. Ann. 191(1971) pp. 326-332.

9. J. R. Holub and J. R. Retherford, "Some curious bases for c_0 and $C(0,1)$," Studia Math. 34(1970) pp. 226-240.

10. W. B. Johnson, H. König, B. Maurey, and J. R. Retherford, "Eigenvalues of p-summing and ℓ_p-type operators in Banach spaces," J.Functional Analysis 32(1979) pp. 353-380.

11. H. König, correspondence, 1981.

12. H. König, s-Zahlen und Eigenwertverteilung von Operatoren in Banachräumen, Habilitation, Friedrich-Wilhelms-Universität, Bonn, 1977.

13. H. Konig, J. R. Retherford, and N. Tomczak-Jaegermann, "On the eigenvalues of (p,2)-summing operators and constants associated with normed spaces," J. Funct. Anal. 37(1980) pp. 88-126.

14. V. B. Lidskij, "Non-selfadjoint operators with a trace," (Russian), Dokl. Acad. Nauk SSSR 125(1959) pp. 485-488.

15. J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces I, Sequence Spaces, Springer-Verlag, Berlin 1977.

16. J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces II, Function Spaces, Springer-Verlag, Berlin, 1978.

17. B. Maurey and G. Pisier, "Séries de variables aléatoires vectorielles indépendantes et propiétés géométriques des espaces de Banach," Studia Math. 58(1976) pp. 45-90.

18. J. S. Morrell and J. R. Retherford, "p-Trivial Banach spaces," Studia Math. 43(1972) pp. 1-25.

19. F. J. Murray and J. von Neumann, "On rings of operators," Ann. of Math. 37(1936) pp. 116-229.

20. J. von Neumann and R. Schatten, "The cross-space of linear transformations II," Ann. of Math. (2) 47(1946) pp. 608-630.

21. A. Pełczyński, "A characterization of Hilbert-Schmidt operators," Studia Math. 28(1966-67) pp. 355-360.

22. A. Pietsch, Operator Ideals, North-Holland, Amsterdam, 1980.

23. A. Pietsch, "Approximation spaces," Forschungsergebnisse Friedrich-Schiller-Universität Jena, N/80/42 (1980).

24. G. Pisier, "Counterexamples to a conjecture of Grothendieck," preprint.

25. G. Pisier, "Some applications of the complex interpolation method to Banach Lattices," J. Analyse Math. 35(1979) pp. 264-281.

26. J. R. Retherford, "Trace," Special Topics of Applied Mathematics, North-Holland, Amsterdam, 1980.

27. H. P. Rosenthal, "On the subspaces of L_p $(p > 2)$ spanned by sequences of independent random variables," Israel J. Math. 8(1970) pp. 273-303.

28. H. H. Schaefer, Topological Vector Spaces, Macmillan, New York, 1966.

29. I. Singer, Bases in Banach Spaces I, Springer, Berlin, 1970.

30. A. Tong, "Diagonal nuclear operators on ℓ_p-spaces," Trans. AMS 143(1969) pp. 235-247.

31. H. Weyl, "Inequalities between the two kinds of eigenvalues of a linear transformation," Proc. NAS(USA) 35 (1949) pp. 408-411.

32. A. Zygmund, Trigonometric Series, Cambridge U. Press, London, 1968.

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